

Building Interest Rate Curves and SABR Model Calibration

by

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Declaration

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Abstract

In this thesis, we first review the traditional pre-credit crunch approach that considers a single curve to consistently price all instruments. We review the theoretical pricing framework and introduce pricing formulas for plain vanilla interest rate derivatives. We then review the curve construction methodologies (bootstrapping and global methods) to build an interest rate curve using the instruments described previously as inputs. Second, we extend this work in the modern post-credit framework. Third, we review the calibration of the SABR model. Finally we present applications that use interest rate curves and SABR model: stripping implied volatilities, transforming the market observed smile (given quotes for standard tenors) to non-standard tenors (or inversely) and calibrating the market volatility smile coherently with the new market evidences.

Keywords: credit crunch/crisis, credit risk, counterparty risk, collateral, CSA, interest rates, negative rates, Libor, Euribor, Eonia, forward curve, discount curve, single-curve, multiple-curve, interest rate derivatives, Deposit, FRA, Futures, OIS, IRS, basis swap, interpolation, global methods, bootstrapping, caps, swaptions, volatility, SABR, calibration.

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How can we forget the Almighty God, the Father of our Lord Jesus Christ in Heaven? We put our trust in You. We praise You, we are grateful, we thank You, we love You.

I will be what people think or say about me, only if I believe it.

Dedications

*To my family, relatives and friends.
A big group with big hearts and strong beliefs.*

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Nomenclature

Notation and Abbreviations

| | |
|--|---|
| ω | Event or outcome |
| Ω | Sample space that consists of all possible outcome |
| \mathcal{F} | Event space |
| \mathbb{P} | Probability measure |
| $(\Omega, \mathcal{F}, \mathbb{P})$ | Probability space |
| σ | σ -algebra |
| T^* | Maximum fixed time horizon for all market activities |
| $\{\mathcal{F}_t\}_{t \in [0, T^*]}$ | Flow of information at time t |
| r_t | Interest rate at time t |
| B_t | Bank Account at time t |
| $P(a, b)$ | Zero Coupon Bond at time a for the maturity b |
| $L(a, b)$ | Simply-compounded spot interest rate (Libor rate) at time a for the maturity b |
| $\delta(a, b)$ | Time interval between time a and b (according to day convention) |
| $F(t, S, T)$ | Simply-compounded forward interest rate |
| K | Fixed rate |
| N | Notional amount |
| $f(a, b)$ | Instantaneous forward interest rate at time a for the maturity b |
| \mathbb{Q}^B | Measure under numéraire B . |
| \mathbb{Q}^T | Measure under numéraire $P(t, \cdot)$ |
| \mathbb{E}^T | Expectation under the T-forward measure |
| $\mathbb{E}^{\mathbb{Q}^B}[\cdot \mathcal{F}_t]$ | Conditional expectation under the measure \mathbb{Q}^B on the \mathcal{F}_t σ -field |
| T | Fixed leg schedule |
| S | Floating leg schedule |
| $S_{\alpha, \beta}(t)$ | Forward Swap Rate |
| \mathcal{C}_s | Yield curve |
| $F_s(t; T_{j-1}, T_j)$ | s -Forward interest rate |
| Cap_t | Cap price at time t |

| | |
|--|---|
| $\Phi(\cdot)$ | Standard Normal cumulative distribution function |
| Floor_t | Floor price at time t |
| $S(t, k, n)$ | ATM strike |
| σ_i | Volatility of the Caplet/Flooret between time interval $[T_{i-1}, T_i]$ |
| $(a - b)^+$ | $\max(a - b, 0)$ |
| $\mathcal{C}_y^P(t_0)$ | Discount yield Curve |
| $\mathcal{C}_y^F(t_0)$ | y -Forward yield Curve |
| r_y | y -Interest rate |
| $B_y(t)$ | Bank Account under y -Interest rate |
| dc_z | Corresponding day count convention for the zero coupon rate |
| \mathbf{T}_y | y Fixed leg schedule |
| $L_{y,j}$ | Spot Libor rate fixed on the market at time T_{j-1} |
| $\mathbf{FRA}_{\text{Std}}$ | Standard Forward Rate Agreement |
| $\text{Depo}(T_j; \mathbf{T}_j)$ | Payoff of the lender in an interest rate Deposit |
| $R_y^{\text{Depo}}(T_0^F; \mathbf{T}_j)$ | y -Simply-compounded Deposit interest rate |
| $\mathbf{FRA}_{\text{Mkt}}$ | Market Forward Rate Agreement |
| $R_y^{\text{Fut}}(t; \mathbf{T})$ | y -Simply-compounded Future interest rate |
| $C_y^{\text{Fut}}(t; T_{j-1})$ | Convexity adjustment |
| $\text{Futures}(t; \mathbf{T})$ | Futures payoff at payment |
| $\mathbf{IRSlet}_{\text{float}}$ | Float coupon payoff Interest Rate Swap |
| $\mathbf{IRSlet}_{\text{fix}}$ | Fixed coupon payoff Interest Rate Swap |
| $A_c(t; \mathbf{S})$ | Annuity Swap discounted with $P_c(t, \cdot)$ |
| R_{on}^{OIS} | Equilibrium OIS rate |
| $\mathcal{C}_c(T_0)$ | OIS Yield Curve at time T_0 |
| SABR | Stochastic Alpha Beta Rho |
| CMS | Constant Maturity Swap |
| sse | Sum Squared Error |
| β | Exponent for the forward rate |
| α | Initial variance |
| ν | Volatility of variance |
| ρ | Correlation between the two Wiener processes |
| K | Strike price |
| F | Forward rate of a log-normal underlying with constant |
| r | Risk-free interest rate |
| σ_B | Black-Scholes constant volatility |
| t_{ex} | Time to maturity |
| σ_j^{mkt} | Market volatilities |

Chapter 1

Introduction

Before the 2007-2008 credit crunch, the single curve approach was predominantly in use for consistently pricing financial instruments. This approach has received less interest in the current literature, perhaps this is because the framework has become obsolete nowadays. Nevertheless, we can refer to [Duffy and Germani \(2013, Chapter 15\)](#). A more detailed literature review can be found in [Chapter 2](#).

In contrast, the multi-curve approach has received a lot of interest in the recent literature. In particular, we can find more information in [Duffy and Germani \(2013, Chapter 16\)](#), [Ametrano and Bianchetti \(2009\)](#), [Ametrano and Bianchetti \(2013\)](#) and others (we refer to [Chapter 3](#) for a more detailed literature review).

The Black-Scholes model is based on the assumption of constant volatility cannot incorporate the volatility smiles usually observed in the markets. Therefore, we must consider alternative stochastic volatility models such as the SABR model. The SABR model was first introduced by [Hagan *et al.* \(2002\)](#). This model has received a lot of attention in the recent literature. Different authors have contributed to its extension and to its improvement. We can cite the works of [Oblój \(2008\)](#) and others (we refer to [Chapter 4](#) for a more detailed literature review).

Problem statement and limitations

Using the SABR model, [Mercurio and Pallavicini \(2006\)](#) proposed a very simple procedure for stripping consistently implied volatilities and CMS adjustments from the market quotes of swaption smiles and CMS swap spreads. Their approach was done in the single-curve framework. We aim to propose an extension of their approach in the multi-curve framework, but we only deal with a method for stripping consistently implied volatilities from the market quotes of swaption smiles. This work is inspired mainly by [Bianchetti and Carlicchi \(2011\)](#) and [Kienitz \(2013\)](#).

To achieve this, we start by reviewing the traditional pre-credit crunch approach that considers a single curve to consistently price all instruments. We then review the curve construction methodologies (bootstrapping and global methods) to build an interest rate curve. Then, we extend this work in the modern post-credit framework. Furthermore, we review the calibration of the SABR model and we highlight the procedure of the calibration after the crisis.

Thesis outline

The thesis is organized as follows. Chapter 2 presents the methodologies (Bootstrapping and Best Fit) for constructing interest rate curves (discounting and forward yield curves), accompanied by their implementation. It reviews the fundamental pricing formulas for plain vanilla interest rate derivatives in the classical framework with no collateral. Chapter 3 gives an overview of a number of changes that have taken place in the financial markets since the credit crunch of 2007. It introduces the use of multiple distinct curves to ensure market coherent estimation of discount factors and of forward rates with different underlying rate tenors. Chapter 4 reviews the calibration of the SABR model for different swaptions. It presents two different methods (with and without refinement), and shows that the SABR model accurately captures the volatility smiles in the markets. Moreover, this chapter reveals the complexity of the market after the credit crunch. Chapter 5 presents applications that use interest rate curves and SABR model such as: stripping implied volatilities, transforming the market observed smile (given quotes for standard tenors) to non-standard tenors (or inversely) and calibrating the market volatility smile coherently with the new market evidences. Finally, the summary and conclusion of the thesis is presented in Chapter 6. The data used in this thesis can be found in Appendix D, also a summary code and more details and proofs of some formulae presented can be found in Appendices A, B, C, E, F, G and H.

Chapter 2

Single Curve

2.1 Introduction

Generally, to trade financial instruments, we need to discount a set of cash flows occurring in the future or to estimate spot rates, forward rates for financial transactions taking place in the future. The best way to achieve all these is to construct an interest rate curve. In the finance market, there are two frameworks for interest rate curve, mainly: Single Curve and Multi or Multiple Curves. The Single Curve represents the traditional or pre-credit crunch approach, which considers a unique curve for both forwarding and discounting. However, after the 2007-2008 credit crunch, a unique curve for both forwarding and discounting is no longer consistent (we will discuss this in Chapter 3). Consequently, new approaches have begun to rise and to be used and the Single Curve framework was superseded by the Multi-Curve framework.

In this chapter, we will provide some basic definitions and necessary concepts that we need later to explain the mechanism of the construction of curves. We assume that there is no default risk in the interbank (default risk and inconsistency between different instruments are ignored in this chapter).

2.2 Definitions and notation

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a filtered probability space describing the market, where:

- Ω is the sample space that consists of all possible outcome $\omega \in \Omega$.
- \mathcal{F} is the event space. It is a σ -algebra consisting of subsets (events) of the sample space $X \subseteq \Omega$.
- \mathbb{P} is the probability measure on the sample Ω . For any event $X \subseteq \Omega$, $\mathbb{P}(X)$ is the probability of X occurring.

Let T^* be the maximum fixed time horizon for all market activities. The filtration $\{\mathcal{F}_t\}_{t \in [0, T^*]}$ (that consists of a family of increasing σ -algebras) represents a flow of information at time t . We have:

- For any $a < b < T^*$ we have $\mathcal{F}_a \subseteq \mathcal{F}_b \subseteq \mathcal{F}_{T^*} \equiv \mathcal{F}$.
- $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Definition 2.1. *A term structure of interest rates is a set of interest rates sorted by time to maturity. The curve shows the relation between the (level of) interest rate (or cost of borrowing) and the time to maturity.*

We can build several types of curves using rates of a different nature, for example zero coupon yield curve, forward rates curve, instantaneous forward curve. Below we provide basic mathematics formulae to deal with these term structures of interest rates.

Definition 2.2 (Bank Account). *Let r_t be a positive stochastic function of time and which models the short term interest rate. At time $t \geq 0$, the value of a bank account is defined by B_t . We assume that the evolution of the bank account satisfies the following differential and initial condition:*

$$dB_t = r_t B_t dt, \quad B_0 = 1. \quad (2.2.1)$$

By simple integration, equation (2.2.1) gives

$$B_t = \exp\left(\int_0^t r_s ds\right) \quad \forall \quad t \in [0, T^*]. \quad (2.2.2)$$

Definition 2.3 (Zero-Coupon Bonds). *A T -maturity zero-coupon bond also called a pure discount bond or a T -bond, is a contract that guarantees its holder the payment of one unit of currency at time T , with no intermediate payments. At time $t \leq T$, the contract value is $P(t, T)$. It follows that $P(T, T) = 1$ for all $T \leq T^*$.*

Using formula (A.0.1), when the bank account B_t is taken as the numéraire (i.e. $N_t = B_t$), then $\mathbb{Q}^N = \mathbb{Q}^B$ and $S(t) = P(t, T)$. It follows that

$$\frac{P(t, T)}{B_t} = \mathbb{E}^{\mathbb{Q}^B} \left[\frac{1}{B_T} \middle| \mathcal{F}_t \right]. \quad (2.2.3)$$

From equation (2.2.3), we have

$$P(t, T) = \mathbb{E}^{\mathbb{Q}^B} \left[\exp\left(-\int_t^T r_u du\right) \middle| \mathcal{F}_t \right] \quad \forall \quad t \in [0, T]. \quad (2.2.4)$$

Definition 2.4 (Simply-compounded spot interest rate). *The simply-compound spot interest rate $L(t, T)$ is the constant rate defined by*

$$L(t, T) = \frac{1 - P(t, T)}{\delta(t, T)P(t, T)} \quad (2.2.5)$$

where $\delta(t, T)$ is the time interval between time t and T (accrual factor according to the day convention chosen¹).

The notation L is motivated by the fact that the market Libor² rates are simply-compounded rates.

Justification

Equation (2.2.5) can be explained as follows: we note $L(t, \tau)$ the Libor rate at time t , of tenor τ ³. At time t , if one lends 1 at Libor, then at time $t + \tau$, one receives back $1 + \delta_L(t, t + \tau)L(t, \tau)$, where $\delta_L(t, t + \tau)$ is the actual time (day count convention) between times t and $t + \tau$.

To avoid arbitrage (ignoring credit issues), we must have

$$\left[1 + \delta_L(t, t + \tau)L(t, \tau)\right]P(t, t + \tau) = 1. \quad (2.2.6)$$

Equation (2.2.5) follows from this last expression.

Definition 2.5 (The T-Forward measure). *The forward martingale measure (or briefly, the T-Forward measure) \mathbb{Q}^T , corresponding to the zero coupon bond $P(t, T)$ maturing at time T is an equivalent probability measure to \mathbb{Q}^B on (Ω, \mathcal{F}_T) , and is defined via the Radon-Nikodým derivative given by*

$$\eta_T = \frac{d\mathbb{Q}^T}{d\mathbb{Q}^B} = \frac{B_T^{-1}}{\mathbb{E}^{\mathbb{Q}^B}[B_T^{-1}]} = \frac{1}{P(0, T)B_T} = \frac{B_0P(T, T)}{P(0, T)B_T}, \quad \mathbb{Q}^B\text{-a.s.} \quad (2.2.7)$$

Proposition 2.6. *The relative prices, for $t \leq U < T$, $P(t, U)/P(t, T)$ are martingales under \mathbb{Q}^T .*

Forward Rates

Forward rates are characterized by three time instants: the current time t at which the rate is considered, the settlement date S and its maturity T with $t < S \leq T$. Forward rates are interest rates applicable to a financial transaction that will take place in the future. We can define a forward rate through a (prototypical) forward-rate agreement (FRA).

¹we refer to Appendix B for more details

²Libor is the abbreviation of London Interbank Offered Rate, it is the average interest rate at which a group of bank in the London market lend money to one another and it is offered in ten major currencies .

³The tenor of interest rate is its maturity period, the period from the point of investment to the time that interest is paid.

Definition 2.7 (FRA). *A FRA is contract in which two counterparties agree to exchange two streams of cash flows in the same currency and in which the notional principal N remains constant over the life of the contract. In the contract, one party pays an interest rate based on the spot rate $L(S, T)$ and receives a fixed rate K .*

Formally, at time T one receives $N\delta(S, T)K$ units of currency and pays the amount $N\delta(S, T)L(S, T)$ (assuming the same day-count conventions for both parties). The value of the contract at time T is therefore:

$$N\delta(S, T)[K - L(S, T)].$$

Using equation (2.2.5), the value of the contract at time T can be written as

$$N\left[\delta(S, T)K - \frac{1}{P(S, T)} + 1\right].$$

The value of the contract at time t , using equation (A.0.2), is

$$P(t, T)\mathbb{E}^{\mathbb{Q}^T}\left[N\left[\delta(S, T)K - \frac{1}{P(S, T)} + 1\right]\middle|\mathcal{F}_t\right].$$

This leads to

$$P(t, T)N[\delta(S, T)K + 1] - P(t, T)N\mathbb{E}^{\mathbb{Q}^T}\left[\frac{1}{P(S, T)}\middle|\mathcal{F}_t\right].$$

Using the fact that $1 = P(S, S)$, we have

$$P(t, T)N[\delta(S, T)K + 1] - P(t, T)N\mathbb{E}^{\mathbb{Q}^T}\left[\frac{P(S, S)}{P(S, T)}\middle|\mathcal{F}_t\right].$$

Using Proposition 2.6, $P(S, S)/P(S, T)$ are martingales under \mathbb{Q}^T , therefore the total value of the contract at time t is

$$\mathbf{FRA}(t, S, T, \delta(S, T), N, K) = N[P(t, T)\delta(S, T)K - P(t, S) + P(t, T)]. \quad (2.2.8)$$

There is one value of K that ensures that the value of the contract at time t is 0. The resulting rate is called the simply-compounded forward rate and which we define below.

Definition 2.8 (Simply-compounded forward interest rate). *At time t , the current time or the date today, we fix two future points S and T such that $t < S \leq T$, where S is called settlement date and T is called time to maturity, the simply-compounded forward interest rate $F(t; T, S)$ for $[S, T]$ contracted at time t , is defined by:*

$$F(t; S, T) = \frac{1}{\delta(S, T)}\left(\frac{P(t, S)}{P(t, T)} - 1\right). \quad (2.2.9)$$

Hence, using equation (2.2.8), the value of the FRA can be written as

$$\mathbf{FRA}(t, S, T, \delta(S, T), N, K) = NP(t, T)\delta(S, T)[K - F(t; S, T)]. \quad (2.2.10)$$

Definition 2.9 (Instantaneous forward interest rate). *The instantaneous forward interest rate, $f(t, T)$, contracted at time t for the maturity $T > t$ is defined by*

$$f(t, T) = \lim_{S \rightarrow T^+} F(t; T, S) = -\frac{\partial \ln P(t, T)}{\partial T}. \quad (2.2.11)$$

The last equality follows from assuming that $\delta(S, T) = T - S$, for small $T - S$

$$\begin{aligned} \lim_{S \rightarrow T^+} F(t; T, S) &= -\lim_{S \rightarrow T^+} \frac{1}{P(t, S)} \frac{P(t, T) - P(t, S)}{S - T} \\ &= -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T} \\ &= -\frac{\partial \ln P(t, T)}{\partial T}. \end{aligned}$$

We consider also the following proposition.

Proposition 2.10. *A simply-compounded forward rate spanning a time interval ending in T is a martingale under the T -forward measure, i.e.*

$$\mathbb{E}^{\mathbb{Q}^T} \left[F(v; S, T) \middle| \mathcal{F}_t \right] = F(t; S, T), \quad 0 \leq t \leq v \leq S < T. \quad (2.2.12)$$

In particular, the forward rate spanning the interval $[S, T]$ is the \mathbb{Q}^T -expectation of the future simply-compounded spot rate at time S for the maturity T , i.e.

$$\mathbb{E}^{\mathbb{Q}^T} \left[L(S, T) \middle| \mathcal{F}_t \right] = F(t; S, T), \quad 0 \leq t \leq S < T. \quad (2.2.13)$$

Definition 2.11 (Swap). *A swap contract is the exchange between two counterparties, with no notional amount exchanged, of cash-flows (interest rates, or currencies). The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated.*

There are many types of swaps. Among them we can cite: Interest Rate Swaps, Currency Swaps, Credit Swaps, Commodity Swaps. Later on, we will discuss some of them, such as: Interest Rate Swap (IRS), Overnight Indexed Swap (OIS), Interest Rate Basis Swap (IRBS).

In a swap, the individual future cash flows that are swapped are called *legs* (there are: *fixed legs and floating legs*) which are calculated over a notional principal amount. In an interest rate swap for instance, usually there is one counterparty which agrees to pay the fixed rate and the other one which agrees

to pay the floating rate (Libor rate for example) over the period, or an adjusted Libor rate (we will assume this in what follows). The fixed or floating rate is multiplied by a notional principal amount and an accrual factor given by the appropriate day count convention.

Definition 2.12 (Over-the-counter market). *Over-the-counter (OTC) market is a decentralized market, without a central physical location, where market participants trade with one another through various communication modes such as the telephone, email and proprietary electronic trading systems.*

Definition 2.13 (Collateral). *Collateral consists of providing an item as a security against the possibility of payment default by the counterparty in a contract.*

Yield curves notation

We denote by \mathcal{C}_y the yield curve defined in a form of continuous term structure or discount factors (or zero coupon bonds)

$$\mathcal{C}_y^P(t_0) = \{T \longrightarrow P_y(t_0, T), T \geq t_0\},$$

or forward rates

$$\mathcal{C}_y^F(t_0) = \{T \longrightarrow F_y(t_0; T, T + y), T \geq t_0\}$$

with t_0 being a reference date of the curves (e.g. settlement date, sport date, or today). The subscript or index y corresponds to tenor, i.e.

$$y \in \{OIS, 1M, 2M, 3M, 6M, 12M\}.$$

2.3 Single curve framework

In this section we introduce the curve construction mechanism to build an interest rate curve. After having introduced the idea of the interest rate curve, we give a basic overview of the instruments used as inputs as well as the main methods used in curve building such as bootstrapping and global methods.

2.3.1 Market instruments selection

There are different types of instruments that can be selected for constructing an interest yield curve, and it is impossible to include all available instruments in the market. Therefore it is important to make a very careful selection of these elements and to minimize the mispricing level of the excluded instruments. In the selection, the priority is given to more liquid⁴ instruments such as:

⁴The features of the liquid asset are: rapidity to be sold, minimal loss of value, any time within the market hours.

Deposits, Forward rate agreements, futures, swaps, etc. We describe these market instruments in detail below.

2.3.1.1 Interest Rate Swaps (IRS)

Interest Rate Swaps are OTC contracts in which two counterparties agree to exchange interest rate cash flows, based on a specified notional amount from a fixed rate to a floating rate (or vice versa) or from one floating rate to another. They are generally used to manage exposure to fluctuations in interest rates.

The illustration is given bellow on Figure 2.1, where the upward pointing arrows are positive cash flows (fixed rate) and the downward pointing arrows are negative cash flows (floating rate). The gray arrows represent the exchanged amount for each period.

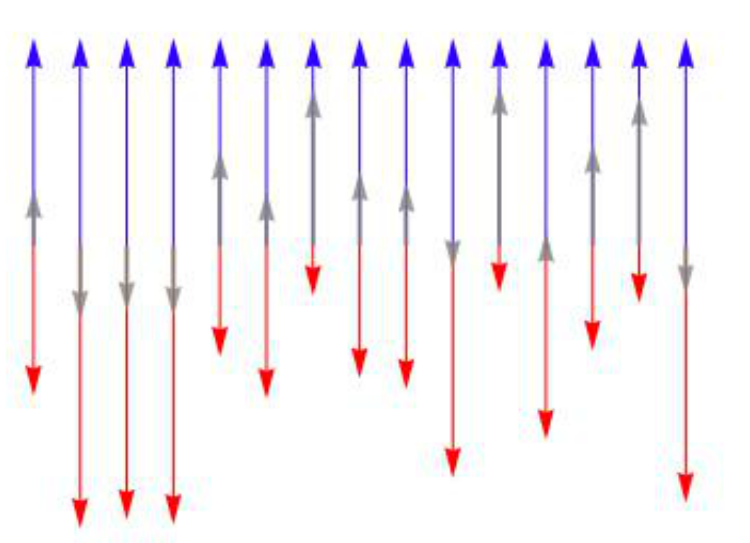


Figure 2.1: Interest Rate Swap cash flows. Blue arrows are cash flows (fixed rate) and red arrows are cash flows (floating rate), the gray arrows are the exchanged amount.

Tables D.5, D.6, D.7, in this order, report the quoted swaps 1M, 3M, 6M Euribor rates. These cash flows are typically tied to a floating Libor rate $L(T_{j-1}, T_j)$ versus a fixed rate K , therefore the IRS is characterised by the following schedule

$$\begin{aligned} \mathbf{T} &= \{T_0, T_1, \dots, T_n\}, & \text{floating leg schedule,} \\ \mathbf{S} &= \{S_0, S_1, \dots, S_m\}, & \text{fixed leg schedule,} \\ S_0 &= T_0, \quad S_n = T_m. \end{aligned}$$

and coupon payoffs are

$$\begin{aligned} \text{IRSlet}_{\text{float}}(T_j; T_{j-1}, T_j, L) &= NL(T_{j-1}, T_j)\delta_L(T_{j-1}, T_j), \quad j = 1, \dots, n \\ \text{IRSlet}_{\text{fix}}(S_i; S_{i-1}, S_i, K) &= NK\delta_K(S_{i-1}, S_i), \quad i = 1 \dots, m \end{aligned} \quad (2.3.1)$$

where $\delta_K(S_{i-1}, S_i)$ is the fix leg time interval between S_{i-1} and S_i and $\delta_L(T_{j-1}, T_j)$ is the floating leg time interval between T_{j-1} and T_j . The coupon payoffs at time t , using equations (A.0.2) and (2.2.13), are given by

$$\begin{aligned} \text{IRSlet}_{\text{float}}(t, T_j; T_{j-1}, T_j, L) &= NP(t; T_j)F(t; T_{j-1}, T_j)\delta_L(T_{j-1}, T_j), \\ \text{IRSlet}_{\text{fix}}(t, S_i; S_{i-1}, S_i, K) &= NK P(t; S_i)\delta_K(S_{i-1}, S_i). \end{aligned}$$

The present value of the fixed leg, at time t , therefore is given by

$$PV_{\text{fixed}}(t) = K \times N \times \sum_{i=1}^m P(t; S_i)\delta_K(S_{i-1}, S_i) \quad (2.3.2)$$

and the present value of the floating leg, at time t , is given by

$$PV_{\text{float}}(t) = N \times \sum_{j=1}^n P(t; T_j)F(t; T_{j-1}, T_j)\delta_L(T_{j-1}, T_j). \quad (2.3.3)$$

Using equation (2.2.9), the sum in equation (2.3.3) can then be simplified as follows

$$\sum_{j=1}^n P(t; T_j)F(t; T_{j-1}, T_j)\delta_L(T_{j-1}, T_j) = P(t; T_0) - P(t; T_n). \quad (2.3.4)$$

However, we will see in the next chapter, this will not be possible in the modern multiple-curve framework. Furthermore, because IRS are traded OTC contracts; formula (A.0.3) will be used instead of formula (A.0.2).

Example 2.14. *Company A and Company B want to borrow a certain amount of money each at the lowest possible cost for a certain period with annual compounding. Company A expects interest rates to decline and wants floating rate borrowing, while Company B expects interest rates to rise and wants to lock-in the fixed rate available to it.*

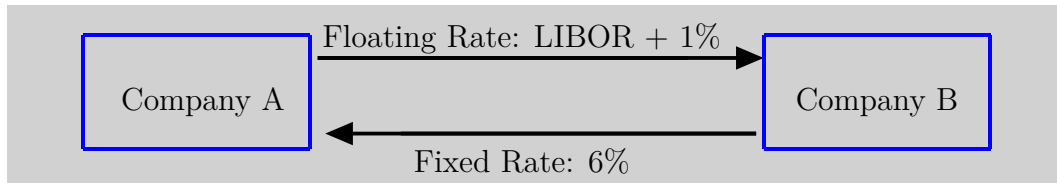


Figure 2.2: Interest rate swap between Companies A and B

Forward Swap Rate

The forward swap rate $S_{\alpha,\beta}(t)$ at time t is the value of K that causes the contract to have zero value at time t , i.e. such that $PV_{\text{float}}(t) = PV_{\text{fixed}}(t)$, we have:

$$S_{\alpha,\beta}(t) = \frac{P(t, S_\alpha) - P(t, S_\beta)}{\sum_{j=\alpha+1}^{\beta} \delta_{j,j-1} P(t, T_j)}. \quad (2.3.5)$$

2.3.1.2 Overnight Indexed Swap (OIS)

An Overnight Indexed swap (OIS) is an agreement between two counterparties to exchange at each payment date or at maturity the difference between fixed rate and floating rate on the nominal amount. The periodic floating rate is equal to the geometric average of an overnight rate over every day of the payment period. For the EUR market the fixing is named EONIA (EUR Overnight Index Average). The principal is not exchanged between counterparties at the end of the trade. The OIS coupon payoffs are given by

$$\begin{aligned} \text{OISlet}_{\text{float}}(T_j; \mathbf{T}_j, R_{on}) &= N R_{on}(T_j; \mathbf{T}_j) \delta_{on}(T_{j-1}, T_j) \quad j = 1, \dots, m, \\ \text{OISlet}_{\text{fix}}(S_i; S_{i-1}, S_i, K) &= \text{IRSlet}_{\text{fix}}(S_i; S_{i-1}, S_i, K) \quad i = 1, \dots, n, \end{aligned} \quad (2.3.6)$$

where $R_{on}(T_j; \mathbf{T}_j)$ is the coupon rate compounded from over night rates over the j -th coupon period (T_{j-1}, T_j) and it is given by

$$R_{on}(T_j; \mathbf{T}_j) := \frac{1}{\delta(T_{j-1}, T_j)} \left[\prod_{k=1}^{n_j} [1 + R(T_{j,k-1}, T_{j,k}) \delta(T_{j,k-1}, T_{j,k})] - 1 \right]$$

where $\mathbf{T}_j = \{T_{j,0}, \dots, T_{j,n_j}\}$, is the sub-schedule for the coupon rate $R(T_j, \mathbf{T}_j)$, and $R(T_{j,k-1}, T_{j,k})$ are the single over night rate spanning the over night time intervals $(T_{j,k-1}, T_{j,k})$. We have also $T_{j,0} = T_{j-1}$, $(T_{j-1}, T_j) = \bigcup_{k=1}^{n_j} (T_{j,k-1}, T_{j,k})$ and $T_{j,n_j} = T_j$, for $j = 1, \dots, n$.

The price and equilibrium rate of the OIS is given, in Appendix E (equations (E.4.9) and (E.4.10)), by

$$\text{OIS}(t; \mathbf{T}, \mathbf{S}, R_{on}^c, K, \omega) = N\omega [R_{on}^{\text{OIS}}(t; \mathbf{T}, \mathbf{S}) - K] A_c(t; \mathbf{S})$$

and

$$\begin{aligned} R_x^{\text{OIS}}(t; \mathbf{T}, \mathbf{S}) &= \frac{\sum_{j=1}^m P_c(t; T_j) R_{on}(t; \mathbf{T}_j) \delta_{on}(T_{j-1}, T_j)}{A_c(t; \mathbf{S})} \\ &= \frac{P_c(t; T_0) - P_c(t; T_m)}{A_c(t; \mathbf{S})} \end{aligned} \quad (2.3.7)$$

where

$$A_c(t; \mathbf{S}) = \sum_{i=1}^n P_c(t; S_i) \delta_K(S_{i-1}; S_i). \quad (2.3.8)$$

In Table D.8, we have a report of the OIS on Eonia from 1W to 60Y which starts at $T_0 = \text{today} + 2$ business and we notice very low and negative quotations for short term OIS.

Considering the OIS schedule $\mathbf{T}_j = \{T_0, \dots, T_j\} = \mathbf{S}_j$, equation (2.3.7) gives

$$R_{on}^{OIS}(T_0; \mathbf{T}_{i-1}) = \frac{P_c(t; T_0) - P_c(t; T_{i-1})}{A_c(T_0; \mathbf{T}_{i-1})}.$$

From equation (2.3.8), we have

$$A_c(T_0; \mathbf{T}_i) = A_c(T_0; \mathbf{T}_{i-1}) + P_c(T_0; T_i)\delta_K(T_{i-1}; T_i).$$

Using the expression of $R_{on}^{OIS}(T_0; \mathbf{T}_i)$, $R_{on}^{OIS}(T_0; \mathbf{T}_{i-1})$ and $A_c(T_0; \mathbf{T}_{i-1})$ above, the discount curve $\mathcal{C}_c^P(T_0)$ at time T_i is given by

$$P_c(T_0; T_i) = \frac{\left[R_{on}^{OIS}(T_0; T_{i-1}) - R_{on}^{OIS}(T_0; T_i) \right] A_c(T_0; T_i) + P_c(T_0; T_{i-1})}{1 + R_{on}^{OIS}(T_0; T_i)\delta_K(T_{i-1}; T_i)}. \quad (2.3.9)$$

Using equations (2.2.9) and (2.3.9), the forward curve $\mathcal{C}_c^F(T_0)$ at time T_i is given by

$$F_c(T_0; T_{i-1}, T_i) = \frac{1}{\delta_{on}(T_{i-1}; T_i)} \left\{ \frac{P_c(T_0; T_i) \left[1 + R_{on}^{OIS}(T_0; T_i)\delta_K(T_{i-1}; T_i) \right]}{\left[R_{on}^{OIS}(T_0; T_{i-1}) - R_{on}^{OIS}(T_0; T_i) \right] A_c(T_0; T_{i-1}) + P_c(T_0; T_{i-1})} - 1 \right\}.$$

2.3.1.3 Deposits

Interest rate deposits (Depos) are standard money market zero coupon contracts. The lender pays the amount N to the borrower at time T_0 and at maturity T_j the borrower pays back to the lender the amount N plus the interest accrued over the period $[T_0, T_j]$ at the simply compounded *Deposit rate* $R_y^{\text{Depo}}(T_0^F; T_j)$, fixed at time $T_0^F \leq T_0$. We have the contract schedule $\{T_0^F, T_0, T_j\}$. For the lender, the payoff at maturity T_j is given by

$$\mathbf{Depo}(T_j; T_j) = N \left[1 + R_y^{\text{Depo}}(T_0^F; T_j)\delta_L(T_0; T_j) \right]. \quad (2.3.10)$$

Since Deposits are not traded on OTC, using equation (A.0.2), the price of the payoff at time t , satisfying $T_0^F \leq t \leq T_j$, is given by

$$\begin{aligned} \mathbf{Depo}(t; T_j) &= P(t; T_j) \mathbb{E}_t^{\mathbb{Q}^{T_j}} \left[\mathbf{Depo}(T_j; \mathbf{T}_j) \right] \\ &= NP(t; T_j) \left[1 + R_y^{\text{Depo}}(T_0^F; T_j)\delta_L(T_0; T_j) \right]. \end{aligned} \quad (2.3.11)$$

Deposits rates are treated as Libor rates, so we have $R_y^{\text{Depo}}(T_0^F; T_j) = L(T_0, T_j)$. Table D.1 shows data on Euro Deposit from 1 day up to 1 year. The discount curve $\mathcal{C}_y^P(T_0)$ at time T_j is obtained using the following relation

$$P_y(T_0, T_j) = \frac{1}{1 + R_y^{\text{Depo}}(t_0; T_j)\delta_L(T_0, T_j)}, \quad T_0 < T_j. \quad (2.3.12)$$

2.3.1.4 Futures

Futures contracts are agreements to buy or sell a stated amount of a security, currency, commodity, or a financial instrument, at a predetermined future date and price. While an option gives the holder the right to buy or sell the underlying asset at expiration, the holder of the Futures contract is obligated to fulfil the terms of his or her contract. Before the crisis, Futures were treated as FRA (Definition 2.7). We will see the change in evaluating Futures in Section 3.3.2.3.

2.3.1.5 Basis Swaps (IRBS)

Interest Rate Basis Swaps are OTC contacts in which two parties exchange two floating rate (with different tenor x and y) payments in the same or different currencies. This is usually done to limit interest-rate risk that a company faces as a result of having differing lending and borrowing rates.

There are two ways of building IRBS instruments: two fixed vs floating IRS, and single IRS floating vs floating plus spread. We only treat the latter way (floating vs floating plus spread), because it is most used.

IRBS as single IRS

Here, the IRBS is a portfolio of a floating vs floating IRS with legs indexed to two different Libors.

$$\begin{aligned}\mathbf{T}_x &= \{T_{x,0}, \dots, T_{x,n_x}\}, & x \text{ leg schedule,} \\ \mathbf{T}_y &= \{T_{y,0}, \dots, T_{y,n_y}\}, & y \text{ leg schedule,} \\ \text{with } T_{x,0} &= T_{y,0}, & T_{x,n_x} = T_{y,n_y}\end{aligned}$$

and coupon payoffs are

$$\begin{aligned}\text{IRBSlet}_x &= NL_x(T_{x,i-1}, T_{x,i})\delta_L(T_{x,i-1}, T_{x,i}) \\ \text{IRBSlet}_y &= N \left[L_y(T_{y,j-1}, T_{y,j}) + \Delta(t; \mathbf{T}_x, \mathbf{T}_y) \right] \delta_L(T_{y,j-1}, T_{y,j})\end{aligned}\tag{2.3.13}$$

with $i = 1, \dots, n_x$; $j = 1, \dots, n_y$; $k = 1, \dots, m$ and where $\Delta(t; \mathbf{T}_x, \mathbf{T}_y)$ in the second leg is a constant basis spread on $L_y(T_{y,j-1}, T_{y,j})$ for maturity $T_{x,n_x} = T_{y,n_y}$. The EUR market quotes standard plain vanilla Basis swap under the form aM vs bM , it is a kind of swap where we have the same fixed legs and floating legs paying Euribor aM and bM . Before the crisis, the basis spread $\Delta(t; \mathbf{T}_x, \mathbf{T}_y)$ was negligible. Hence

$$\text{IRBSlet}_x \simeq \text{IRBSlet}_y.$$

Therefore, for instance, for a IRBS receiving Euribor yM and paying Euribor $3M$ for maturity T_j , we have

$$R_x^{\text{IRS}}(t, \mathbf{T}, \mathbf{S}) \simeq R_{3M}^{\text{IRS}}(t, \mathbf{T}, \mathbf{S}).\tag{2.3.14}$$

Interpolation

Interpolation is a method of constructing new data points within the range of a discrete set of known data points. In other words, it is a process of approximating the value of a function $y(t)$ (satisfying $y(\tau_j) = y_j$) between two points at which it has prescribed values.

Interpolation is very important in yield curve construction and determines some characteristics of the curve. A lot of risk and money can be hidden behind the interpolation method. There are many choices of interpolation function, however we are interested in interpolations that preserve arbitrage-free conditions, localness (a change in an input, affects the shape of the curve only locally), smoothness, positivity and stability (a change in an input, does not affect the entire shape of the curve) of forward rates. A typical headache for an interest rate trader is to choose between forward curve smoothness and bump hedge localness. Also we cannot use more than one method simultaneously (because each method satisfies specific requirement).

Interpolation, as we will see in Example 2.4.5, can be used in two phases. First, directly on market quotes to complete missing information. Second, internally to return discount factors for time intervals not directly covered by information of interest rates (the interaction between these two phases is crucial).

In Appendix C, we present the interpolation methods that are used in this work. More details about interpolation can be found in Hagan and West (2008) or Duffy and Germani (2013).

2.4 Curve construction mechanism

There are two main methods for curve construction starting from market data: traditional bootstrapping method and the global method. In both cases the curve building process should be calibrated to a set of quotes by solving the equations that set the theoretical values equal to the market values. Also, in both cases, we underline that interpolation is needed to complete missing data through the process of calibration.

2.4.1 Bootstrapping method

The traditional bootstrapping method consists in solving the equations that set the theoretical values equal to market values sequentially (Example 2.4.5). We assume that different instruments are associated with different dates, so that, for instance, we cannot calibrate both on a 6m deposit and a 6m swap since they would be associated with the same calibration date. Starting from shorter maturities we progress sequentially to longer maturities. In the process missing information can be retrieved using interpolation. Because the process is done

sequentially, for this reason only interpolation that preserves the localness should be used (for example, the linear on the logarithm of discount factor interpolation method).

2.4.2 Best Fit method

Curve fitting is the process of constructing a curve that has the best fit to a series of data points, possibly subject to constraints. Curve fitting can involve either interpolation, where an exact fit to the data is required, or smoothing, in which a “smooth” function is constructed that approximately fits the data (for this end we may use Levenberg-Marquardt nonlinear least squares solver algorithm).

2.4.3 Single curve market instruments selection

The single curve was usually constructed based on the selection of the following market instruments⁵ (more information can be found in common textbooks such as: [Hull \(2009\)](#), [Rebonato and Rebonato \(1998\)](#), [Chibane *et al.* \(2009\)](#)).

1. Deposit contracts, covering the window from today up to 1Y.
2. FRA contracts, covering the window from 1M up to 2Y.
3. Futures contracts, covering the window from 3M up to 2Y and more.
4. IRS contracts, covering the window from 2Y-3Y up to 30Y.

The instruments cited above are not homogeneous in the underlying rate (they admit underlying interest rates with mixed tenors).

2.4.4 Single curve bootstrapping approach

The pre-crisis standard market practice, which was based on the single-curve approach (and that can be found for instance in: [Ron \(2000\)](#), [Ametrano and Bianchetti \(2009\)](#), [Bianchetti \(2008\)](#), [Hagan and West \(2006\)](#), [Ametrano \(2011\)](#), [Hagan and West \(2008\)](#) and [Andersen \(2007\)](#)), can be summarised in the following steps:

1. Interbank credit/liquidity issues do not matter for pricing, Libors are good proxy for risk free rates, Basis Swap spreads are negligible (and not taken into consideration).
2. The collateral does not matter for pricing, Libor discounting is adopted.

⁵also called blocks

3. Select *one* finite set of the most convenient vanilla instrument traded on the market with increasing maturities.
4. Construct *one* yield curve by using the selected instruments (as in Section 2.4.3).
5. Compute *on the same curve* FRA rates, discounts factors by using formulae presented in Section 2.3.1 above.
6. If necessary, compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the *same* set of vanillas.

We emphasize that, in this framework, on a given currency, a unique yield curve is built and used to price any interest rate derivative. It goes the same way to suppose that there exists a unique underlying short rate process that is able to model and explain the whole term structure of interest rates for all tenors. In addition, the prices of a derivative are calculated relatively to a set of plain vanillas quoted on the market. Finally given the fact that discount factors and forward rates are obtained by interpolation, it follows in general that, there may exist arbitrage possibilities.

General settings

The *reference date* for the yield curve can be: today, spot date⁶ or in principal any business day after today. The reference date of the EUR market, except ON (OverNight i.e. between today and tomorrow) and TN (Tomorrow Next i.e. between Tomorrow and Next day) Deposit contracts, is $t_0 = \text{spot date}$. The vector of all the dates of the curve from the reference date up to any maturity is called the *Time grid or pillars or also knots*. The parameter which defines the reference currency of the yield curve corresponding to the currency of the instruments is called *Calendar*.

TARGET Business Day refers to a day on which the Trans-European Automated Real-time Gross Settlement Express Transfer (TARGET) System, or any successor thereto, is operating credit or transfer instructions with respect of payments in Euro. *Business Day Convention* is a procedure used for adjusting payment dates in response to days that are not TARGET Business Days. *Following Business Day Convention* is a procedure in which payment days that fall on a Holiday or Saturday or a Sunday roll forward to the next TARGET Business Day. On the other hand *Modified Following Business Day Convention* is a procedure in which payment days that fall on a Holiday or Saturday or a Sunday roll forward to the next TARGET Business Day, unless that day falls in the next calendar month, in which case the payment day rolls backward to the immediately preceding TARGET Business Day.

⁶ spot date of a transaction is the normal settlement day when the transaction is done today.

2.4.5 Yield curve bootstrapping example

We present here an example of bootstrapping a yield curve from data market. We use the theory presented above in a more pragmatic manner. In Section 2.4.3, we have seen how to select markets instruments. Using data in Tables D.1, D.2, D.4 and D.5, we select markets instruments that we report in Table 2.1 below. In Section 2.4.4, we have presented the steps used in the bootstrapping method. The idea is to build up sequentially the yield curve from shorter maturities to longer maturities (**15y** in this example).

The spot date t_0 is 13 December, 2012 (13/12/12). The chosen day-count convention is the Actual/360, hence

$$\delta(T, S) = \frac{\text{actual number of days between } T \text{ and } S}{360}.$$

| Deposit (%) | | FRA (%) | | Futures | | Swaps (%) | |
|-------------|-------|-------------------|-------|-----------|---------|-----------|-------|
| SN | 0.040 | FRA 2×5 | 0.141 | 18 Sep 13 | 99.8725 | 2y | 0.324 |
| 1w | 0.070 | FRA 4×10 | 0.256 | 18 Dec 13 | 99.8425 | 3y | 0.424 |
| 1m | 0.110 | | | 19 Mar 14 | 99.8025 | 4y | 0.576 |
| 2m | 0.140 | | | 18 Jun 14 | 99.7425 | 5y | 0.762 |
| 3m | 0.180 | | | | | 7y | 1.135 |
| 6m | 0.320 | | | | | 10y | 1.584 |
| | | | | | | 15y | 2.037 |

Table 2.1: Data selected from Appendix D (D.1)

- The first column in Table 2.1 contains the Deposit rates for maturities

$$\{T_1, \dots, T_6\} = \{14/12/12, 20/12/12, 14/01/13, 13/02/13, 13/03/13, 13/06/13\}.$$

Therefore, we have 1, 7, 32, 62, 90 and 182 days to maturity, respectively. The discount factor, using equation (2.3.12), is

$$P(t_0, T_i) = \frac{1}{1 + L(t_0, T_i)\delta(t_0, T_i)}.$$

for $i = 1, \dots, 6$.

- The second column in Table 2.1 contains the FRA rates for maturities

$$\{V_1, V_2\} = \{13/05/13, 15/10/13\}.$$

Note that the FRA 2×5 starts the 13/02/13 and matures the 13/05/13 and the FRA 4×10 starts the 15/04/13 and matures the 15/10/13.

Therefore, we have 89 and 184 days to maturity, respectively. Then using equation (2.2.9), we have

$$P(t_0, 13/05/13) = \frac{P(t_0, 13/02/13)}{1 + \delta(13/02/13, 13/05/13)F(t_0, 13/02/13, 13/05/13)}.$$

$P(t_0, 13/02/13) = P(t_0, T_4)$ is known from the previous block. Similarly, we have

$$P(t_0, 15/10/13) = \frac{P(t_0, 15/04/13)}{1 + \delta(15/04/13, 15/10/13)F(t_0, 15/04/13, 15/10/13)}.$$

Here, however, $P(t_0, 15/04/13)$ is unknown. **This is where interpolation comes in.** We can interpolate $P(t_0, 15/04/13)$ between $P(t_0, T_5)$ and $P(t_0, T_6)$ from the previous block.

- The futures are quoted as futures price (third column in Table 2.1). For settlement day U_i , we can find futures rate by using the relation

$$100(1 - F_F(t_0; U_i, U_{i+1}))$$

where $F_F(t_0; U_i, U_{i+1})$ is the futures rate for period $[U_i, U_{i+1}]$ prevailing at t_0 , and

$$\{U_1, \dots, U_5\} = \{18/09/13, 18/12/13, 19/03/14, 18/06/14, 17/09/14\}$$

hence $\delta(U_i, U_{i+1}) = 91/360$. We treat futures rates as if they were simple FRA rates (previous block), that is, we set

$$F(t_0; U_i, U_{i+1}) = F_F(t_0; U_i, U_{i+1}).$$

Then using equation (2.2.9), we have

$$P(t_0, U_{i+1}) = \frac{P(t_0, U_i)}{1 + \delta(U_i, U_{i+1})F(t_0, U_i, U_{i+1})}.$$

By this formula, we are able to calculate the discount factor $P(t_0, U_i)$ for $i = 2, 3, 4, 5$. However, we need to calculate $P(t_0, U_1)$ first. Once again, interpolation is needed. We can interpolate $P(t_0, U_1)$ between $P(t_0, V_1)$ and $P(t_0, V_2)$ from the previous block.

The linear interpolation of the discount factors is given by:

$$P(0, T) = \frac{T_k - T}{T_k - T_{k-1}} P(0, T_{k-1}) + \frac{T - T_{k-1}}{T_k - T_{k-1}} P(0, T_k),$$

for $T_{k-1} \leq T \leq T_k$. While the linear interpolation of the log discount factors is:

$$\log P(0, T) = \frac{T_k - T}{T_k - T_{k-1}} \log P(0, T_{k-1}) + \frac{T - T_{k-1}}{T_k - T_{k-1}} \log P(0, T_k),$$

for $T_{k-1} \leq T \leq T_k$.

- The fourth column in Table 2.1 contains the swaps rates (semi-annual) for maturities

$$\{S_1, \dots, S_{30}\} = \left\{ \begin{array}{l} 13/06/13, \quad 13/12/13, \quad 13/06/14, \quad 15/12/14, \quad 15/06/15 \\ 14/12/15, \quad 13/06/16, \quad 13/12/16, \quad 13/06/17, \quad 13/12/17 \\ 13/06/18, \quad 13/12/18, \quad 13/06/19, \quad 13/12/19, \quad 15/06/20 \\ 14/12/20, \quad 14/06/21, \quad 13/12/21, \quad 13/06/22, \quad 13/12/22 \\ 13/06/23, \quad 13/12/23, \quad 13/06/24, \quad 13/12/24, \quad 13/06/25 \\ 15/12/25, \quad 15/06/26, \quad 14/12/26, \quad 14/06/27, \quad 13/12/27 \end{array} \right\}$$

The swap rate at t_0 with maturity S_n is given by

$$R_{\text{swap}}(t_0, S_n) = \frac{P(t_0, S_0) - P(t_0, S_n)}{\sum_{i=1}^n \delta(S_i, S_{i+1})P(t_0, S_i)}, \quad (S_0 := t_0). \quad (2.4.1)$$

From the data we have $R_{\text{swap}}(t_0, S_i)$ for $i = 4, 6, 8, 10, 14, 20, 30$. Once more, we obtain $P(t_0, S_1)$, $P(t_0, S_2)$ (and hence $R_{\text{swap}}(t_0, S_1)$, $R_{\text{swap}}(t_0, S_2)$) by interpolation using previous blocks. All remaining swap rates are obtained through interpolation. For maturity S_3 for instance, using linear interpolation, we have

$$R_{\text{swap}}(t_0, S_3) = \frac{1}{2} (R_{\text{swap}}(t_0, S_2) + R_{\text{swap}}(t_0, S_4)).$$

Using equation (2.4.1), we get

$$P(t_0, S_n) = \frac{P(t_0, S_0) - R_{\text{swap}}(t_0, S_n) \sum_{i=1}^{n-1} \delta(S_{i-1}, S_i) P(t_0, S_i)}{1 + R_{\text{swap}}(t_0, S_n) \delta(S_{n-1}, S_n)}.$$

This last formula gives $P(t_0, S_n)$ recursively for $n = 3, \dots, 30$.

Once we have calculated all the discount factors P , we are able to calculate all the forward rates F (by using equation (2.2.8)) as well. We can then plot, the numerical results over time. The forward rate curve produced by this method may be in some case wobbly, this implies that there may be some problems with the method.

Problems with the Bootstrapping Method

An approximation is used when treating futures as if they were FRAs. In fact an adjustment, called the convexity adjustment, is required to convert Futures prices to equivalent FRAs (as we will see in equation (3.3.14)).

Some interpolations are required for the missing data, these interpolation methods produce characteristic problems with forward rates calculated from the curve. Also, because this bootstrapping method works up from rates of nearer maturity to rates of further maturity, a slight change in a nearer maturity rate can cause variations or oscillations farther up the forward rate curve.

A poor bootstrapping method produces a characteristic symptom of a saw-tooth structure in the forward rate curve. This can be seen at the transition maturities between different instruments and at knot points of an inappropriate scheme.

An alternative method is to use Best Fit Method that would estimate a smooth yield curve parametrically from the market rates.

2.4.6 Implementation

We apply here the methodologies illustrated in the previous sections to the concrete EUR market case found in (Ametrano and Bianchetti, 2009), (Ametrano and Bianchetti, 2013). We only report the yield curves: C_{3M}^F and C_{6M}^F . The numerical results have been obtained using (Duffy and Germani, 2013, Chapter 15) and QuantLib framework⁷. The yield curves reported here are the final result of a complex chain of choices. Many alternatives choices are possible.

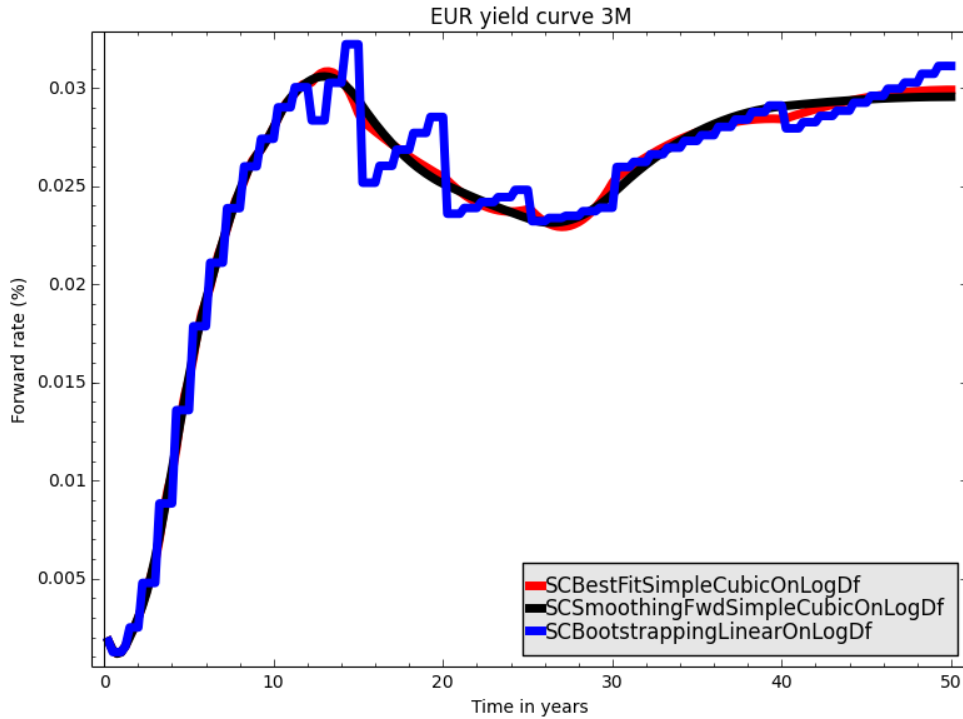


Figure 2.3: 3M-Forward curve up to 50 years. Red: Single Curve using Best Fit with Simple Cubic interpolation on log of discount factor, black: Single Curve using Best Fit smoothing with Simple Cubic interpolation on log of discount factor, blue: Single Curve using Bootstrapping with Linear interpolation on log of discount factor.

⁷More details can be found in Appendix H.

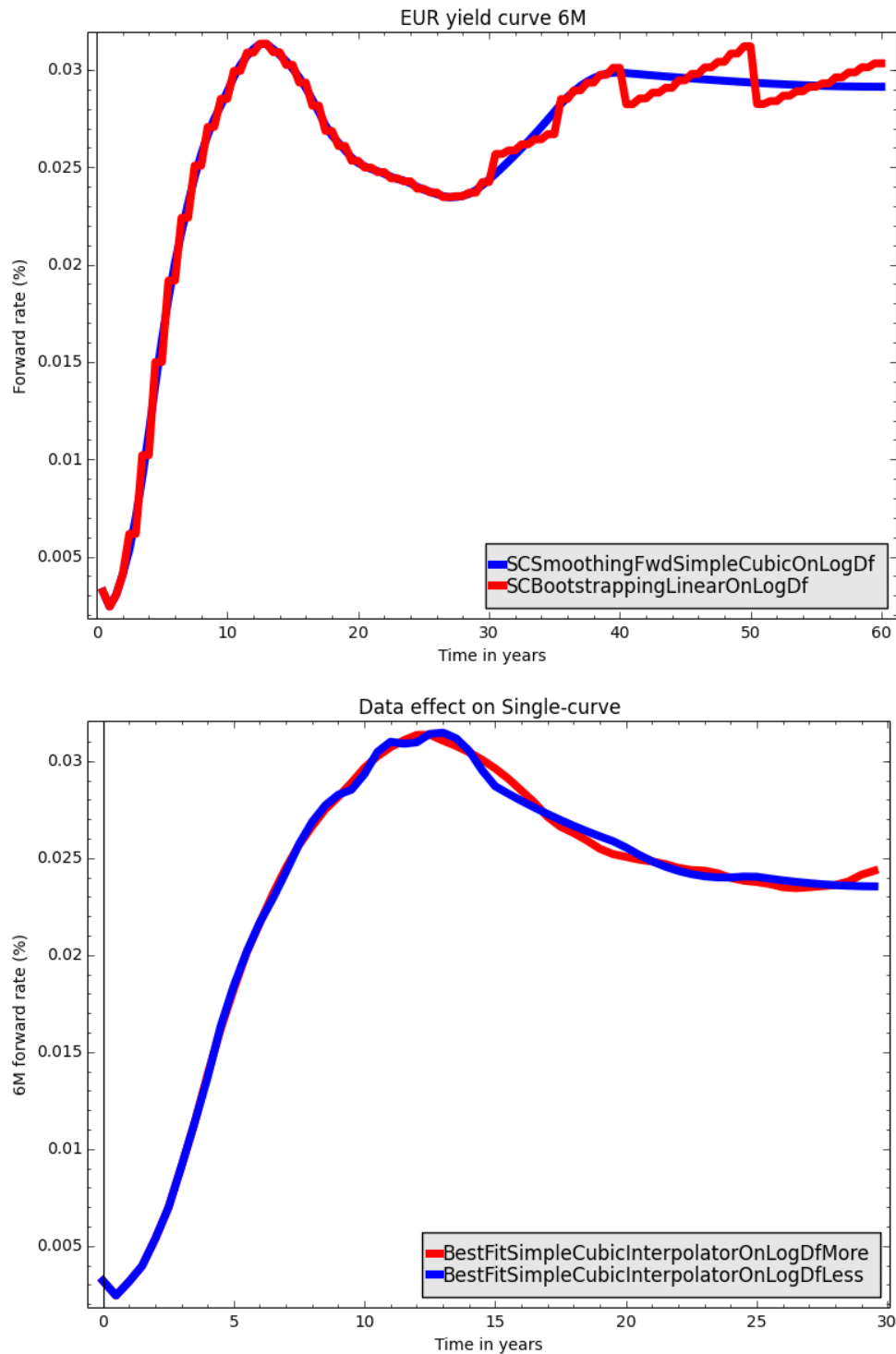


Figure 2.4: Top panel: 6M-Forward curve up to 60 years. Red: Single Curve using Bootstrapping with Linear interpolation on log of discount factor, blue: Single Curve using Best Fit smoothing with Simple Cubic interpolation on log of discount factor. Bottom panel: effect of data on the forward single curve. Red: Bootstrapping with Linear interpolation on log of discount factor using more data, blue: Bootstrapping with Linear interpolation on log of discount factor using less data.

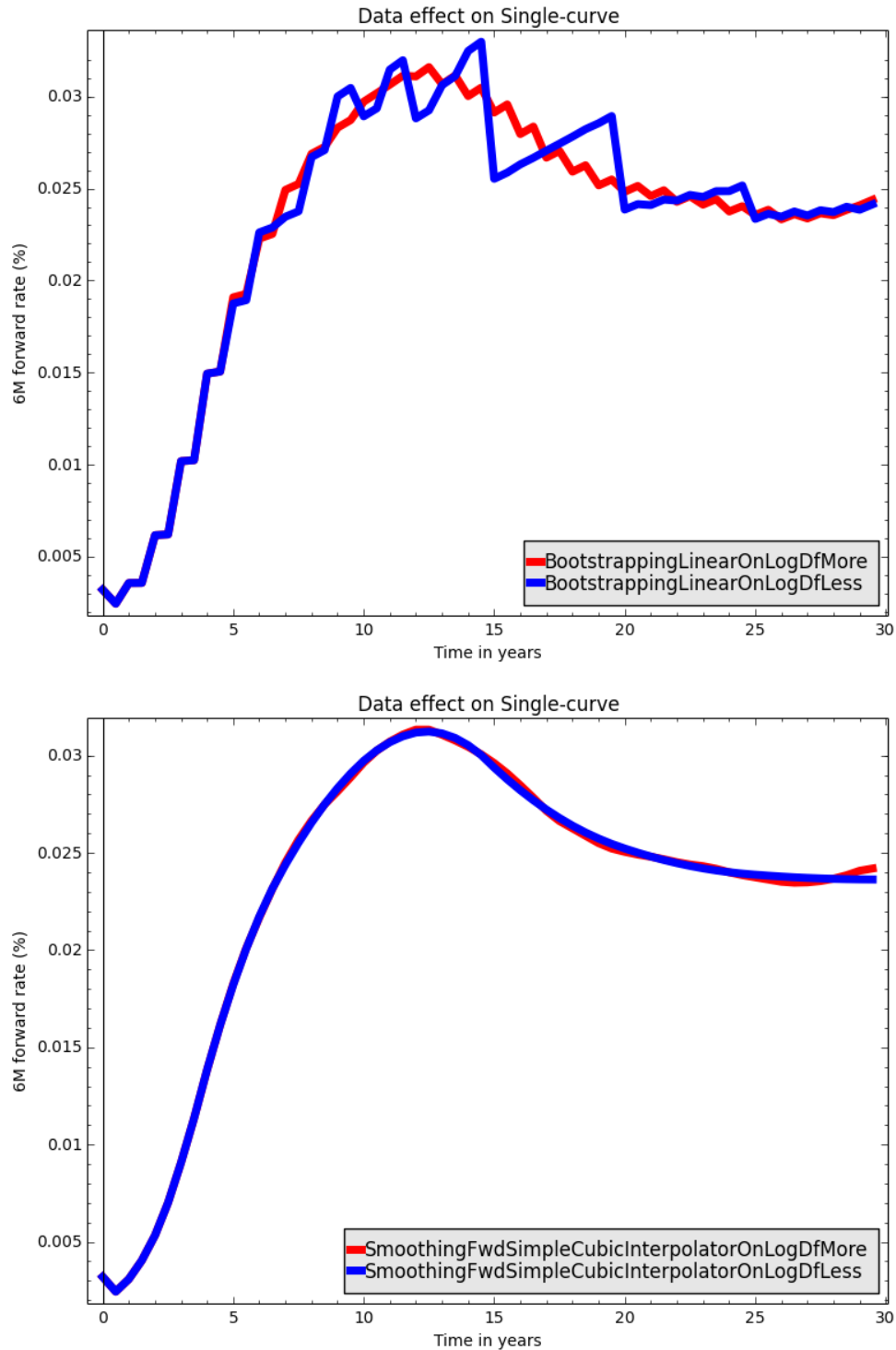


Figure 2.5: Top panel: effect of data on the forward single curve. Red: Best Fit with Linear interpolation on log of discount factor using more data, blue: Best Fit with Linear interpolation on log of discount factor using less data. Bottom panel: effect of data on the forward single curve. Red: Smoothing Forward with Simple Cubic interpolation on log of discount factor using more data, blue: Smoothing Forward with Simple Cubic interpolation on log of discount factor using less data.

Figure 2.3 is the 3M-Forward curve up to 50 years. Figure 2.4 (top panel) shows the 6M-Forward curve up to 60 years. “**SCBootstrappingLinearOnLogDf**” stands for Single Curve using Bootstrapping methodology with Linear interpolation on log of discount factor, “**SCBestFitSimpleCubicOnLogDf**” stands for Single Curve using Best Fit methodology with Simple Cubic interpolation on log of discount factor (the curve is not necessarily smooth) and “**SCSmoothingFwdSimpleCubicOnLogDf**” stands for Single Curve using Best Fit methodology with Simple Cubic interpolation on log of discount factor (the curve is required to be smooth on forward rate). The difference between the two methodologies (Bootstrapping and Best Fit) and the impact of the chosen interpolation method can be seen from the figure. It can be seen from the figures that there is no much difference between yield curves C_{3M}^F and C_{6M}^F .

Figure 2.4 (bottom panel) and Figure 2.5 are the plots of 6M-Forward curve. “**BestFitSimpleCubicInterpolatorOnLogDfMore**” stands for Best Fit methodology with Linear interpolation on log of discount factor using more data, “**BestFitSimpleCubicInterpolatorOnLogDfLess**” stands for Best Fit methodology with Linear interpolation on log of discount factor using less data. “**BootstrappingLinearOnLogDfMore**” stands for Bootstrapping methodology with Linear interpolation on log of discount factor using more data, “**BootstrappingLinearOnLogDfLess**” stands for Bootstrapping methodology with Linear interpolation on log of discount factor using less data. “**SmoothingFwdSimpleCubicOnLogDfMore**” stands for Best Fit methodology with Simple Cubic interpolation on log of discount factor using more data, “**SmoothingFwdSimpleCubicOnLogDfLess**” stands for Best Fit methodology with Simple Cubic interpolation on log of discount factor using less data.

As expected, when market quotes are rare, there is a higher impact from the interpolation scheme on forward rates shape, also from the methodologies. Other strategies allow us to mitigate the impact from the interpolation scheme, for instance we build the discount curve using linear interpolation on the discount factors and we change interpolation for forward rates.

2.5 Options caps, floors and swaptions

We introduce here options on caps, floors and swaptions, which are over-the-counter (OTC) contracts and known as plain-vanilla (or standard) interest-rate options.

Definition 2.15. A **cap** is an OTC contract by which the seller agrees to payoff a positive amount to the buyer of the contract if the reference rate exceeds a prespecified level called the exercise rate of the cap on given future dates. The seller of a **floor** agrees to pay a positive amount to the buyer of the contract if the reference rate falls below the exercise rate on some future dates.

We note some terms: the reference rate is an interest-rate index based, for example, on *Libor*, swap rates from which the contractual payments are determined; the exercise rate or strike rate is a fixed rate determined at the origin of the contract; the settlement frequency refers to the frequency with which the reference rate is compared to the exercise rate; the starting date is the date when the protection of caps and floors begins.

Let us consider a cap with a nominal amount N , an exercise rate K , based upon an underlying rate $L(T_{i-1}, T_i)$ which covers a period from T_{i-1} to T_i and with the schedule $\{T_0, T_1, \dots, T_n\}$. T_0 is the starting date of the cap and $T_n - T_0$ expressed in years is the maturity of the cap. On each date payment T_i , the cap holder receives a cash flow C_i , given by:

$$C_i = N\delta(T_{i-1}, T_i)(L(T_{i-1}, T_i) - K)^+. \quad (2.5.1)$$

C_i is a call option on $L(T_{i-1}, T_i)$ observed on date T_{i-1} with a payoff occurring on date T_i . The cap is a portfolio of n such options. The n call options of the cap are known as the **caplets**.

The cap price at date t in the Black (1976) model is given by

$$\begin{aligned} \text{Cap}_t = \sum_{i=1}^n \text{Caplet}_t^i &= \sum_{i=1}^n N\delta(T_{i-1}, T_i)P(t, T_i) \left[F(t; T_{i-1}, T_i)\Phi(d_i) \right. \\ &\quad \left. - K\Phi(d_i - \sigma_i\sqrt{T_{i-1} - t}) \right]. \end{aligned} \quad (2.5.2)$$

Where

$$d_i = \frac{\log\left(\frac{F(t; T_{i-1}, T_i)}{K}\right) + \frac{\sigma_i^2(T_{i-1} - t)}{2}}{\sigma_i\sqrt{T_{i-1} - t}}$$

and where σ_i is the volatility of a caplet over $[T_{i-1}, T_i]$, and Φ is the standard Normal cumulative distribution function.

Let us now consider a floor with the same characteristics. The floor holder gets on each date T_i

$$F_i = N\delta(T_{i-1}, T_i)(K - L(T_{i-1}, T_i))^+. \quad (2.5.3)$$

F_i is a put option on $L(T_{i-1}, T_i)$ observed on date T_{i-1} with a payoff occurring on date T_i . The floor is a portfolio of n such options. The n put options of the floor are known as the **floorlets**.

The floor price at date t in the Black (1976) model is given by

$$\begin{aligned} \text{Floor}_t = \sum_{i=1}^n \text{Floorlet}_t^i &= \sum_{i=1}^n N\delta(T_{i-1}, T_i)P(t, T_i) \left[-F(t; T_{i-1}, T_i)\Phi(-d_i) \right. \\ &\quad \left. + K\Phi(-d_i + \sigma_i\sqrt{T_{i-1} - t}) \right]. \end{aligned} \quad (2.5.4)$$

2.5.1 Cap and floor at the money strike

A cap or a floor is considered ATM (at the money) if the strike K is equal to the forward swap rate calculated according to the cap or floor conventions. At time T_i , we define ϕ by

$$\phi(T_i) = \sum_{i=1}^n (C_i - F_i).$$

Using equations (2.5.1) and (2.5.3), we have

$$\phi(T_i) = \sum_{i=1}^n N\delta(T_{i-1}, T_i) (L(T_{i-1}, T_i) - K).$$

Using equations (A.0.2) and (2.2.13), we have

$$\phi(t, T_i) = \sum_{i=1}^n NP(t, T_i)\delta(T_{i-1}, T_i) (F(t; T_{i-1}, T_i) - K).$$

Hence,

$$\text{Cap} - \text{Floor} = \sum_{i=1}^n NP(t, T_i)\delta(T_{i-1}, T_i) (F(t; T_{i-1}, T_i) - K).$$

Therefore the ATM strike is given by

$$S = \frac{\sum_{i=1}^n \delta(T_{i-1}, T_i) P(t, T_i) F(t; T_{i-1}, T_i)}{\sum_{i=1}^n \delta(T_{i-1}, T_i) P(t, T_i)}. \quad (2.5.5)$$

In the single-curve framework, using equation (2.2.8), the ATM strike is given by

$$S(t, k, n) = \frac{P(t, T_k) - P(t, T_n)}{\sum_{i=k+1}^n \delta(T_{i-1}, T_i) P(t, T_i)}. \quad (2.5.6)$$

Definition 2.16. *A swaption is an option granting its owner the right but not the obligation to enter into an underlying swap. Although options can be traded on a variety of swaps, the term “swaption” typically refers to options on interest rate swaps.*

There are two types of swaption contracts:

- *A payer swaption gives the owner of the swaption the right to enter into a swap where they pay the fixed leg and receive the floating leg.*
- *A receiver swaption gives the owner of the swaption the right to enter into a swap in which they will receive the fixed leg, and pay the floating leg.*

The Black formula for a payer and a receiver swaption are

$$Payer(t) = \sum_{i=k+1}^n N\delta(T_{i-1}, T_i)P(t, T_i) \left[S(t, k, n)\Phi(d_i) - K\Phi(d_i - \sigma_i\sqrt{T_{i-1} - t}) \right] \quad (2.5.7)$$

and

$$Receiver(t) = \sum_{i=k+1}^n N\delta(T_{i-1}, T_i)P(t, T_i) \left[K\Phi(-d_i + \sigma_i\sqrt{T_{i-1} - t}) - S(t, k, n)\Phi(-d_i) \right] \quad (2.5.8)$$

where

$$d_i = \frac{\log\left(\frac{S(t, k, n)}{K}\right) + \frac{\sigma_i^2(T_k - t)}{2}}{\sigma_i\sqrt{T_k - t}}.$$

Finally, we define an at the money (ATM) payer or receiver swaption as a swaption having the strike K , as defined in formulae (2.5.7) and (2.5.8), equal to at the money swap as shown in formula (2.5.6).

2.6 Summary and conclusion

In this chapter we have described each market instrument used for the construction of an interest curve in the pre-crisis framework. We have illustrated methodologies (Bootstrapping and Best Fit) for constructing both discounting and FRA yield curves, consistently with market instruments. We have presented the implementation of both methodologies, we have focused on interest rate swap given that this instrument plays a relevant role in the curve building process. We have reviewed the fundamental pricing formulas for plain vanilla interest rate derivatives in the classical framework with no collateral.

Chapter 3

Multi-Curves

3.1 Introduction

The credit crunch crisis of summer 2007 has revealed a large Basis Swap spread¹ between single-currency interest rate instruments of different tenors, as a consequence just one single curve is not appropriate for market coherent estimation of forward rates of different tenors, such as 1, 3, 6, 12 months.

During the 2007 crisis it became clear that these no-arbitrage assumptions (equations (2.2.5), (2.2.6) and (2.2.9)) could break down due to counterparty² and liquidity risk³, for example. Moreover, it became more important to collateralise OTC deals in order to reduce the risk involved in bilateral transactions. The multi-curve framework was introduced precisely for the purpose of dealing with collateralised derivatives and with the new behaviour of the forward rates. The traditional framework, using the same curve for discounting and for estimating forward rates, was not flexible enough to capture these features; some “new” formulae are required.

In this chapter, we present current challenges about curve building across the market. We stress that there is default risk in the interbank. Also, we use the methodologies for building interest rate yield curves described in Chapter 2 in the new framework.

The present of curves

Media around February 2013 have revealed the problem of a “Libor Scandal” (also called “the crime of the century”), which was a revelation of the dishonest practices connected to the Libor. Consequently, the Libor has lost its credibility for being considered as a true proxy for borrowing costs or for fund-

¹We refer to 2.3.1.5 for more details.

²Counterparty risk refers to the risk that the other party in an agreement will default.

³Liquidity risk is the risk that a given security or asset cannot be traded quickly enough in the market to prevent a loss (or make the required profit).

ing costs. The banks members were falsely inflating or deflating their rates in order to profit from trades, or to give the impression that they were more creditworthy than they were.

Following the credit crisis 2007-2008, new counterparty risk mitigation techniques based on collateral agreements are widely used (even before the revelation of the “Libor Scandal”). The appropriate rate at which cash flows should be discounted when valuing a collateralized trade is the rate at which collateral earns interest.

3.2 Pricing valuation after the credit crunch

We highlight some reasons that have led to the requirement for a new pricing framework for interest rates derivatives valuation.

3.2.1 Impact of collateralisation

Standard derivatives pricing theory relies on the assumption that one can borrow and lend at a unique risk-free rate. However, the situation is rather different these days; as a consequence of the 2007-2008 global financial crisis, historically stable relationships between government rates, Libor rates, etc, have broken down.

In order to mitigate credit risk among dealers, agreements (based on the so-called credit support annex (CSA) to the International Swaps and Derivatives Association (ISDA) master agreement) have been put in place to collateralise mutual exposures. Therefore Collateral could then be thought of as an essentially risk-free investment. The rate on collateral (such as the fed funds rate for dollar transactions, Eonia for Euro, etc) is usually set to be a proxy of a risk-free rate. In addition, the pricing of non-collateralised derivatives needs to be adjusted accordingly, as compared to the collateralised version.

- The first adjustment is to use different discounting rates for CSA and non-CSA versions of the same derivative.
- The second adjustment is a convexity, or quanto, adjustment and affects forward curves as they turn out to depend on collateralisation used.
- The third adjustment that may be required is to volatility information used for options, in particular, the volatility smile changes depending on collateral.

We denote the short rate at time t by $r_c(t)$; c here stands for CSA, as we assume this is the agreed overnight rate paid on the safest available collateral (cash) among dealers under CSA. The corresponding discount factor is $P_c(t, T)$.

3.2.2 OIS discounting

An Overnight Indexed Swap (OIS) is a fixed/floating interest rate swap. The floating leg is tied to a published overnight rate index (EONIA). The counterparties agree to exchange (at the repayment date) the difference between the agreed fixed rate and the interest accrued from the geometric average of the daily floating overnight index rate over the time period based on the agreed notional amount.

OISs popularity have increased since the 2007/2008 financial crisis for many reasons: they are liquid and credit-efficient derivatives for all major currencies, Libor-based instruments often failed to capture movements in policy rates, they can be used to hedge against moves in overnight interest rates.

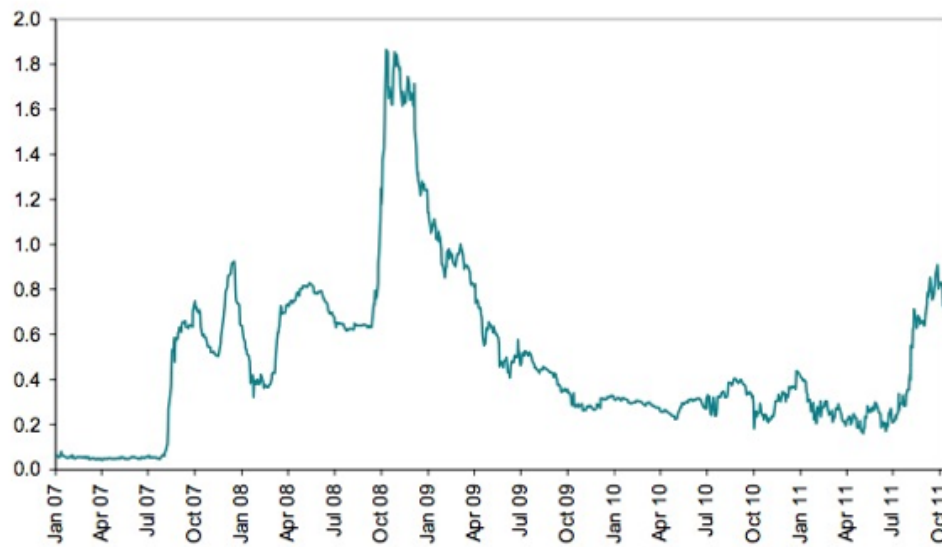
Collateralisation impacts the price of a derivative instrument and this is one of the reason why the multicurve framework is needed. Consider, for example, a swap exchanging a fixed rate against a Libor rate. Before the crisis, the cash flows were discounted using the discount factors derived from the Libor swap curve. However under collateralisation, the curve for discounting cash flows of a swap should depend on how the swap is collateralised or funded. In a case of bilateral contract with a poorly rated counterparty, we have an impact of the counterparty risk on the valuation of the derivative. Therefore it is not correct to use OIS discounting.

3.2.3 Basis

As it can be seen on Figure 3.1, before the financial crisis the 3m Euribor-OIS spread had an average value of 8 basis points (average from May 2000 to July 2007). So there was not a significant difference in pricing swap discounting with the Euribor curve rather than the OIS curve. From August 2007 to August 2008 the average spread was about 66 basis points. The crisis introduced a widening of the Euribor-OIS spread as a consequence of deteriorating bank credit quality and fear of uncertainty.

Figure 7: Funding costs are elevated

3m Euribor/EONIA spread (%)



Source: Datastream

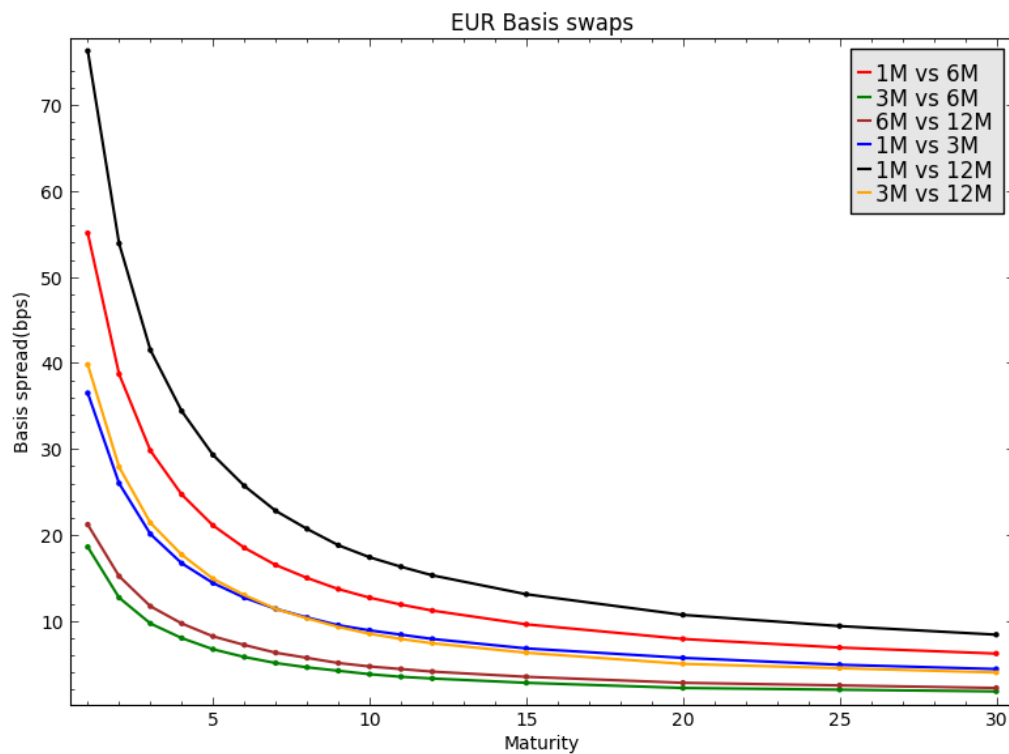


Figure 3.1: 3m Euribor-Eonia, Basis Swap between different tenor.

3.3 Multi curve framework

Given the notation described above (Chapter 2), below we describe the evolution of the financial market.

3.3.1 Multi curve market instruments selection

Multi Curves are usually constructed based on the selection of the following market instruments⁴ (this way of selecting market instruments can be found in common textbooks such as: Hull (2009), Rebonato and Rebonato (1998))

1. Deposit contracts, covering the window from today up to 1Y.
2. FRA contracts, covering the window from 1M up to 2Y.
3. Futures contracts, covering the window from 3M up to 2Y and more.
4. IRS contracts, covering the window from 2Y-3Y up to 30Y.
5. IRBS contracts, covering the window up to 60Y.

In contrast to Section 2.4.3, here the instruments are not homogeneous in the underlying rate (they do not admit underlying interest rates with mixed tenors). We also note the presence of IRBS contracts.

3.3.2 The multi-curve approach

Some of the weaknesses of the single curve approach presented in Chapter 2 are:

- The basis swap spread can no longer be neglected due to the crisis (as pointed out in Section 3.2.3 above).
- As a consequence of the credit crunch, mixing different underlying rate tenors is no longer possible.
- Libor discounting was used⁵.

In order to remedy these weaknesses, the market has adopted a new framework which the summary of the procedure is given in the following steps:

1. Interbank/liquidity issues do matter for pricing, Libors risky rates, Basis Swap spreads are no longer negligible (and taken into consideration).
2. The collateral does matter for pricing, OIS discounting is adopted (Section 3.2.2).

⁴also called blocks

⁵This is important to avoid arbitrage opportunities.

3. Decide the appropriate discounting rate of derivatives to be priced, then select the corresponding market instruments and build **one single discounting curve** \mathcal{C}_d using the classical single-curve methodologies described in Chapter 2.
4. Select multiple separated sets of vanilla interest rate instruments traded on the market with increasing maturities, each set of **homogeneous** in the underlying rate (e.g. $1M, 2M, 3M, 6M, 12M$ tenors).
5. Build **multiple distinct forward curves** $\mathcal{C}_{1M}, \mathcal{C}_{3M}, \mathcal{C}_{6M}, \mathcal{C}_{12M}$ using the selected sets of vanilla interest rate market instruments plus their bootstrapping rules⁶ and the unique discount curve \mathcal{C}_d .
6. If necessary, compute the delta sensitivity⁷ with respect to the market pillars of **each** yield curve $\mathcal{C}_d, \mathcal{C}_{1M}, \mathcal{C}_{3M}, \mathcal{C}_{6M}, \mathcal{C}_{12M}$ and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the **corresponding** set of vanillas.

We note that, this new approach has been introduced to remedy the impact of the credit crisis as far as curve building is concerned, however practitioners should be aware of some technical challenges such as:

- It can be inferred from the procedure of multiple curve building that the choice of the discounting curve is crucial, consequently, this should be done with more care.
- Multiples curves require many quotations available on the market such as: Deposits, Futures, Swaps, Basis Swaps, FRA (we need data for each tenor).
- Some interpolation methods may not produced smooth forward curves.
- All the theory about curve building must be reviewed, tested and (in some case) may be adapted to the new markets realities. This is not an easy task for practitioners.

3.3.2.1 Interest Rates

Let $L_y(T_{j-1}, T_j) := L_{y,j}$ be the spot Libor rate fixed on the market at time T_{j-1} and spanning the interval $[T_{j-1}, T_j]$ where y is the rate tenor. These rates are mostly “risky” since they are traded with unsecured financial transactions. Depending on the type of transaction, tenor and possible collateral, the degree of risk may be small or significant. Forward Libor rates are quoted in the

⁶Rules such as using the most liquid instruments, avoiding selecting instruments with same maturities and so on.

⁷We do not present this point in this work.

OTC derivatives market in terms of equilibrium rates of FRA (Forward Rate Agreement) contracts. For simplicity we assume that both Libor and fixed rates have the simply compounded annual convention and they share the same year fraction δ_L . Based on the discussion given in Definition 2.7, the standard FRA payoff at cash flow date T_j is given by

$$\mathbf{FRA}_{\text{Std}}(T_j; \mathbf{T}, L_{y,j}, K, \omega) = \omega N \left[L_y(T_{j-1}, T_j) - K \right] \delta_L(T_{j-1}, T_j). \quad (3.3.1)$$

The FRA contract given in equation (3.3.1) is called “standard” or “textbook” FRA and differs from the “market” FRA quoted in the market (it is developed in Section 3.3.2.2). Since FRA are traded on OTC, using equation (A.0.3), the price of the payoff at time t is given by

$$\begin{aligned} \mathbf{FRA}_{\text{Std}}(t; \mathbf{T}, L_{y,j}, K, \omega) &= P_c(t; T_j) \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\mathbf{FRA}_{\text{Std}}(T_j; \mathbf{T}, L_{y,j}, K, \omega) \right] \\ &= \omega N P_c(t; T_j) \left\{ \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[L_y(T_{j-1}, T_j) \right] - K \right\} \delta_L(T_{j-1}, T_j) \\ &= \omega N P_c(t; T_j) \left[F_{y,j}(t) - K \right] \delta_L(T_{j-1}, T_j) \end{aligned} \quad (3.3.2)$$

where

$$F_{y,j}(t) := \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[L_y(T_{j-1}, T_j) \right]. \quad (3.3.3)$$

The FRA contact equilibrium rate is the value of the fixed rate K which is a solution of the equation

$$\mathbf{FRA}_{\text{Std}}(t; \mathbf{T}, L_{y,j}, K, \omega) = 0.$$

We get

$$K =: R_{y,\text{Std}}^{\text{FRA}}(t, T_{j-1}, T_j) = F_{y,j}(t) = \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[L_y(T_{j-1}, T_j) \right]. \quad (3.3.4)$$

3.3.2.2 Forward Rate Agreements (FRAs)

Forward Rate Agreements are OTC contracts between parties that determine the rate of interest, or the currency exchange rate, to be paid or received on an obligation starting at some point (date) in the future.

The actual FRA traded on the market, has a different payoff, such that, at payment date T_{j-1} (not T_j), we have

$$\begin{aligned} \mathbf{FRA}_{\text{Mkt}}(T_{j-1}; \mathbf{T}, L_{x,j}, K, \omega) &= N \omega \frac{[L_x(T_{j-1}, T_j) - K] \delta_F(T_{j-1}, T_j)}{1 + L_x(T_{j-1}, T_j) \delta_F(T_{j-1}, T_j)} \\ &= \frac{\mathbf{FRA}_{\text{Std}}(T_j; \mathbf{T}, L_x, K, \omega)}{1 + L_x(T_{j-1}, T_j) \delta_F(T_{j-1}, T_j)}. \end{aligned} \quad (3.3.5)$$

Since FRAs are traded OTC between collateralized counterparties, we may apply the pricing under collateral approach using equation (A.0.3). The price and the equilibrium rate, derived in Appendix E (equations (E.1.4) and (E.1.5)), are given by

$$\mathbf{FRA}_{Mkt}(t; \mathbf{T}, K) = NP_c(t; T_{j-1}) \left[1 - \frac{1 + K\delta_x(T_{j-1}, T_j)}{1 + F_{x,j}(t)\delta_x(T_{j-1}, T_j)} e^{C_{c,x}^{FRA}(t; T_{j-1})} \right] \quad (3.3.6)$$

and

$$R_{x,Mkt}^{FRA}(t, \mathbf{T}) = \frac{1}{\delta_x(T_{j-1}, T_j)} \left\{ \left[1 + F_{x,j}(t)\delta_x(T_{j-1}, T_j) \right] e^{-C_{c,x}^{FRA}(t; T_{j-1})} - 1 \right\}, \quad (3.3.7)$$

where $e^{C_{c,x}^{FRA}(t; T_{j-1})}$ is the *convexity adjustment*, depending on the particular model adopted for the dynamics of the rates $F_{d,j}(t)$ and $F_{x,j}(t)$.

The actual size of the convexity adjustment, even for long maturities, is below 1 bp (more details can be found in (Mercurio, 2010)). It follows that, in any practical situation, we can neglect the convexity adjustment and use the classical pricing expressions, hence

$$\mathbf{FRA}_{Mkt}(t; \mathbf{T}, L_{x,j}, K, \omega) \simeq \mathbf{FRA}_{Std}(t; \mathbf{T}, L_{x,j}, K, \omega).$$

It follows that

$$\mathbf{FRA}_{Mkt}(t; \mathbf{T}, L_{x,j}, K, \omega) \simeq \omega NP_c(t; T_j) \left[F_{x,j}(t) - K \right] \delta_L(T_{j-1}, T_j) \quad (3.3.8)$$

and

$$R_{x,Mkt}^{FRA}(t, \mathbf{T}) \simeq R_{x,Std}^{FRA}(t, T_{j-1}, T_j) = F_{x,j}(t) = \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[L_x(T_{j-1}, T_j) \right]. \quad (3.3.9)$$

FRAs are presented in the form $a \times b$ FRA which mean: a Effective date from now, b termination date from now, and $b - a$ months Deposits. From Table D.2, for instance the 2×8 FRA starts at $T = 2$ months and matures at $T = 8$ months, it is a six months (6M) Deposit starting two months forward. We note that, the FRA dates concatenate exactly: the 2×8 FRA matures at $T = 6M$ where the following 8×14 starts.

Market FRAs provide direct empirical evidence that a single curve cannot be used to estimate forward rates with different tenors. In the Table D.2 for instance, we can see that the level of the market 2×5 FRA3M (spanning from 13th February to 13th May, $\delta_L(2 \times 5) = 0.24722$) was $R_{3M,Mkt}^{FRA}(t_0, 2 \times 5) = 0.141\%$, the level of the market 5×8 FRA3M (spanning from 13th May to 13th August, $\delta_L(5 \times 8) = 0.25555$) was $R_{3M,Mkt}^{FRA}(t_0, 5 \times 8) = 0.124\%$. By compounding these two rates, the level of the implied 2×8 FRA6M (spanning from 13th February to 13th August, $\delta_L(2 \times 8) = 0.50278$) would be

$$1 + R_{6M,Implied}^{FRA}(t_0, 2 \times 8)\delta_L(2 \times 8) = \left[1 + R_{3M,Mkt}^{FRA}(t_0, 2 \times 5)\delta_L(2 \times 5) \right] \left[1 + R_{3M,Mkt}^{FRA}(t_0, 5 \times 8)\delta_L(5 \times 8) \right].$$

It follows that

$$R_{6M,Implied}^{FRA}(t_0, 2 \times 8) = \frac{\left[1 + R_{3M,Mkt}^{FRA}(t_0, 2 \times 5)\delta_L(2 \times 5)\right] \left[1 + R_{3M,Mkt}^{FRA}(t_0, 5 \times 8)\delta_L(5 \times 8)\right] - 1}{\delta_L(2 \times 8)}.$$

Hence

$$R_{6M,Implied}^{FRA}(t_0, 2 \times 8) = 0.1324\%.$$

However, the market quote for 2×8 FRA6M was $R_{6M,Mkt}^{FRA}(t_0, 2 \times 8) = 0.272\%$, about 13.96 basis point larger.

The discount factor at time T_j is given by

$$P_y(T_0, T_j) = \frac{P_y(T_0, T_{j-1})}{1 + R_{y,Mkt}^{FRA}(t_0; \mathbf{T}_j)\delta_L(T_{j-1}, T_j)}. \quad (3.3.10)$$

It can be seen that

$$\lim_{T_{j-1} \rightarrow T_0} R_{y,Mkt}^{FRA}(t; \mathbf{T}_j) = R_y^{Depo}(t_0, T_j). \quad (3.3.11)$$

3.3.2.3 Futures

Futures were introduced in Chapter 2, Section 2.3.1.4. Interest Rate Futures are equivalent to the OTC FRAs. The Futures payoff at payment date T_{j-1} is given, from a point of view of the counterparty paying the floating rate, by

$$\mathbf{Futures}(T_{j-1}; \mathbf{T}) = N \left[1 - L_y(T_{j-1}, T_j) \right]. \quad (3.3.12)$$

The Futures rate at time $t < T_{j-1}$, derived in Appendix E (equations (E.2.2) and (E.2.3)), are given by

$$\mathbf{Futures}(t; \mathbf{T}) = NP_c(t; T_{i-1}) \left\{ 1 - \frac{\mathbb{E}_t^{Q_c^{T_i}} \left[\left(1 + L_c(T_{i-1}, T_i) \delta_c(T_{i-1}, T_i) \right) L_y(T_{i-1}, T_i) \right]}{1 + F_{c,i}(t) \delta_c(T_{i-1}, T_i)} \right\} \quad (3.3.13)$$

and

$$R_y^{Fut}(t; \mathbf{T}) = F_{y,j}(t) + C_y^{Fut}(t; T_{j-1}) \quad (3.3.14)$$

where $R_y^{Fut}(t; \mathbf{T})$ is the rate implied by Futures price and its corresponding FRA rate is $F_{y,j}(t)$.

A specific model is needed to compute the convexity adjustment to find the Futures rate. For instance the mean reversion and the Volatility of the Hull & White model were 3% and 0.3526% respectively for Futures3M at 11

Dec. 2012. It can be seen, on Table D.4, that futures are quoted in terms of prices instead of rates. We have the following relation

$$\mathbf{Futures}(t; \mathbf{T}) = 100 \times \left[1 - R_y^{Fut}(t; \mathbf{T}) \right]. \quad (3.3.15)$$

Using equations (2.2.9) and (3.3.14), the discount curve $\mathcal{C}_y^P(t_0)$ at time T_j is given by

$$P_y(T_0, T_j) = \frac{P_y(T_0, T_{j-1})}{1 + [R_y^{Fut}(t_0; \mathbf{T}_j) - C_y^{Fut}(t_0; T_j)]\delta_L(T_j, T_j)}. \quad (3.3.16)$$

If $\mathcal{C}_y(T_{j-1})$ is known, the expression above can be used to bootstrap $\mathcal{C}_y(T_j)$.

3.3.2.4 Interest Rate Swaps (IRS)

IRS are traded OTC between collateralized counterparties, we may use the formula in equation (A.0.3). The IRS price, rate and annuity, derived in Appendix E (equations (E.3.5), (E.3.6) and (E.3.7)), are given by

$$\mathbf{IRS}(t; \mathbf{T}, \mathbf{S}, L_y, K, \omega) = N\omega [R_y^{\text{IRS}}(t; \mathbf{T}, \mathbf{S}) - K] A_c(t; \mathbf{S}),$$

$$R_y^{\text{IRS}}(t; \mathbf{T}, \mathbf{S}) = \frac{\sum_{j=1}^m P_c(t; T_j) F_{y,j}(t) \delta_L(T_{j-1}, T_j)}{A_c(t; \mathbf{S})} \quad (3.3.17)$$

and

$$A_c(t; \mathbf{S}) = \sum_{i=1}^n P_c(t; S_i) \delta_K(S_{i-1}, S_i). \quad (3.3.18)$$

We underscore that the classical telescopic property (as in equation (2.3.4)), such that

$$\sum_{j=1}^m P_c(t; T_j) F_{x,j}(t) \delta_L(T_{j-1}, T_j) = P_c(t; T_0) - P_c(t; T_m) \quad (3.3.19)$$

does not hold any longer, because in the modern multiple-curve framework the discount rate P_c and the forward rate $F_{x,j}$ belong to two different yield curves.

From equation (3.3.17), the forward swap rate $S_{\alpha,\beta}(t)$ at time t is given by

$$S_{\alpha,\beta}(t) = \frac{\sum_{\beta=1}^j P_c(t; T_\beta) F_{y,\beta}(t) \delta_L(T_{\beta-1}, T_\beta)}{\sum_{j=\alpha+1}^\beta P_c(t; S_j) \delta_K(S_{j-1}, S_j)}. \quad (3.3.20)$$

From equation (3.3.17) the forward curve $\mathcal{C}_y^F(T_0)$ at time $T_j = S_i$ is given by

$$F_{y,j}(T_0) = \frac{R_y^{\text{IRS}}(T_0; T_j) A_c(T_0; T_j) - \sum_{\beta=1}^{j-1} P_c(T_0; T_\beta) F_{y,\beta}(T_0) \delta_L(T_{\beta-1}, T_\beta)}{P_c(T_0; T_j) \delta_L(T_{j-1}, T_j)}. \quad (3.3.21)$$

Using equation (2.2.9), we have

$$\frac{P_y(T_0; T_{j-1})}{P_y(T_0; T_j)} = 1 + F_{y,j}(T_0) \delta_L(T_{j-1}; T_j).$$

Using equation (3.3.21) above, we get

$$\frac{P_y(T_0; T_{j-1})}{P_y(T_0; T_j)} = 1 + \frac{R_y^{\text{IRS}}(T_0; T_j) A_c(T_0; T_j) - \sum_{\beta=1}^{j-1} P_c(T_0; T_\beta) F_{y,\beta}(T_0) \delta_L(T_{\beta-1}, T_\beta)}{P_c(T_0; T_j)}.$$

Hence

$$P_y(T_0; T_j) = \frac{P_c(T_0; T_j) P_y(T_0; T_{j-1})}{P_c(T_0; T_j) + R_y^{\text{IRS}}(T_0; T_j) A_c(T_0; T_j) - \sum_{\beta=1}^{j-1} P_c(T_0; T_\beta) F_{y,\beta}(T_0) \delta_L(T_{\beta-1}, T_\beta)}. \quad (3.3.22)$$

The equations (3.3.21) and (3.3.22) can be used to bootstrap the yield curve \mathcal{C}_y at point $T_j = S_i$ if we already know the curve at points $\mathbf{T} = \{T_1, \dots, T_{j-1}\}$ and $\mathbf{S} = \{S_1, \dots, S_{i-1}\}$. Some points between $P_y(t_0, T_{j-1})$ and $P_y(t_0, T_j)$ in equation (3.3.22) can be unknown; we use interpolation for this end.

As an example, the bootstrap of Euribor6M curve \mathcal{C}_{6M} from 19Y to 20Y knots, we have $R_{6M}^{\text{IRS}}(T_0, T_{20}) = 2.187\%$ in Table D.5, is given by

$$F_{6M,20Y}(T_0) = \frac{R_{6M}^{\text{IRS}}(20Y) A_c(20Y) - \sum_{\beta=1}^{j-1} P_c(T_0; T_\beta) F_{y,\beta}(T_0) \delta_L(T_{\beta-1}, T_\beta)}{P_c(20Y) \delta_L(T_{j-1}, T_j)} \quad (3.3.23)$$

and

$$P_{6M}(T_{20Y}) = \frac{P_c(20Y) P_{6M}(19.5Y)}{P_c(20Y) + R_{6M}^{\text{IRS}}(20Y) A_c(20Y) - \sum_{\beta=1}^{j-1} P_c(T_0; T_\beta) F_{y,\beta}(T_0) \delta_L(T_{\beta-1}, T_\beta)}. \quad (3.3.24)$$

Euribor6M being semi-annual, so T_{20} correspond to S_{40} , so we have $\mathbf{T} = \{T_1, \dots, T_{20}\}$, $\mathbf{S} = \{S_1, \dots, S_{40}\}$, $T_{19} = S_{38} = 19Y$, $S_{39} = 19.5Y$, $T_{20} = S_{40} = 20Y$. We have to interpolate $P_{6M}(S_{39})$ (unknown) in equation (3.3.24). We can see (as we have seen in Section 2.4.5), the bootstrapping procedure contains the interpolation.

3.3.2.5 Basis Swaps (IRBS)

Coupon payoffs were already defined in Chapter 2 (Section 2.3.1.5) as

$$\begin{aligned} \text{IRBSlet}_x &= N L_x(T_{x,i-1}, T_{x,i}) \delta_L(T_{x,i-1}, T_{x,i}) \\ \text{IRBSlet}_y &= N \left[L_y(T_{y,j-1}, T_{y,j}) + \Delta(t; \mathbf{T}_x, \mathbf{T}_y) \right] \delta_L(T_{y,j-1}, T_{y,j}) \end{aligned}$$

with $i = 1, \dots, n_x$; $j = 1, \dots, n_y$; and where $\Delta(t; \mathbf{T}_x, \mathbf{T}_y)$ in the second leg is a constant basis spread on $L_y(T_{y,j-1}, T_{y,j})$ for maturity $T_{x,n_x} = T_{x,n_y}$. The equilibrium basis spread of the IRBS is given by, in Appendix E (equation (E.5.2)),

$$\Delta(t; \mathbf{T}_x, \mathbf{T}_y) = \frac{\text{IRS}_{x,\text{float}}(t; \mathbf{T}_x, L_x) - \text{IRS}_{y,\text{float}}(t; \mathbf{T}_y, L_y)}{N A_c(t; \mathbf{T}_y)}. \quad (3.3.25)$$

After the crisis, the basis spread $\Delta(t; \mathbf{T}_x, \mathbf{T}_y)$ is no longer negligible. Therefore, for instance, for a IRBS receiving Euribor xM and paying Euribor $3M$ for maturity T_j , we have

$$R_x^{\text{IRS}}(t, \mathbf{T}, \mathbf{S}) = R_{3M}^{\text{IRS}}(t, \mathbf{T}, \mathbf{S}) + \Delta(t, \mathbf{T}_x, \mathbf{T}_{3M}, \mathbf{S}). \quad (3.3.26)$$

Table D.9 presents possible basis combinations: $1M$ vs $6M$, $3M$ vs $6M$ and $6M$ vs $12M$. Basis Swaps allow to find non-quoted Swaps (usually for long maturities) on Euribor $1M$, $3M$, and $12M$.

3.3.3 Implementation

We implement here the yield curves in the Multi Curve framework.

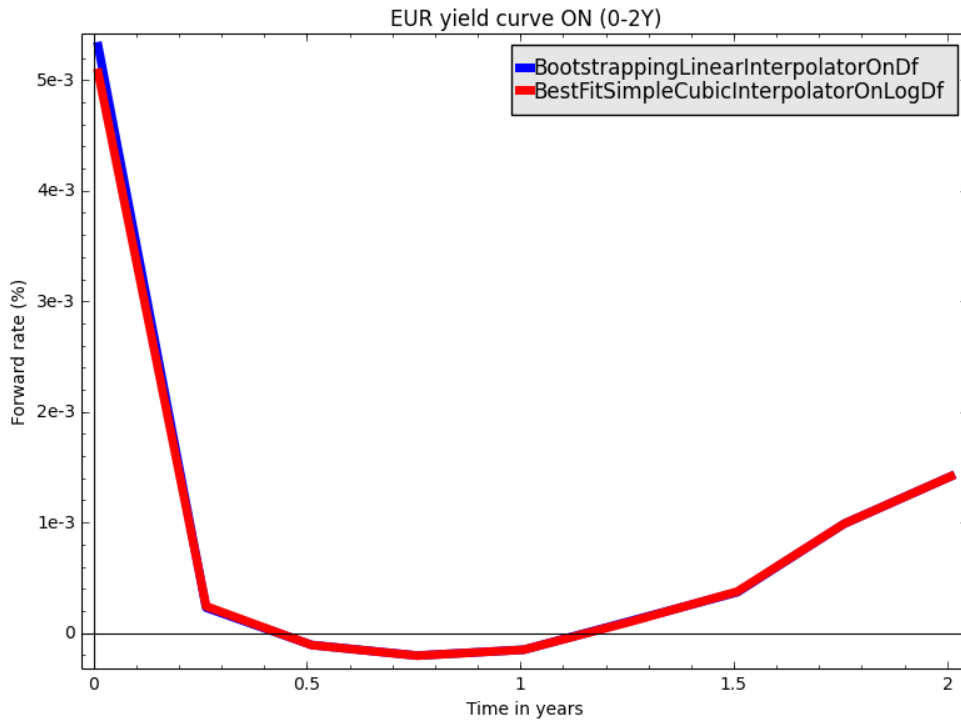


Figure 3.2: EONIA 3M-Forward curve up to 2 years. Blue: Single Curve using Bootstrapping with Linear interpolation on log of discount factor, red: Single Curve using Best Fit with Simple Cubic interpolation on log of discount factor.

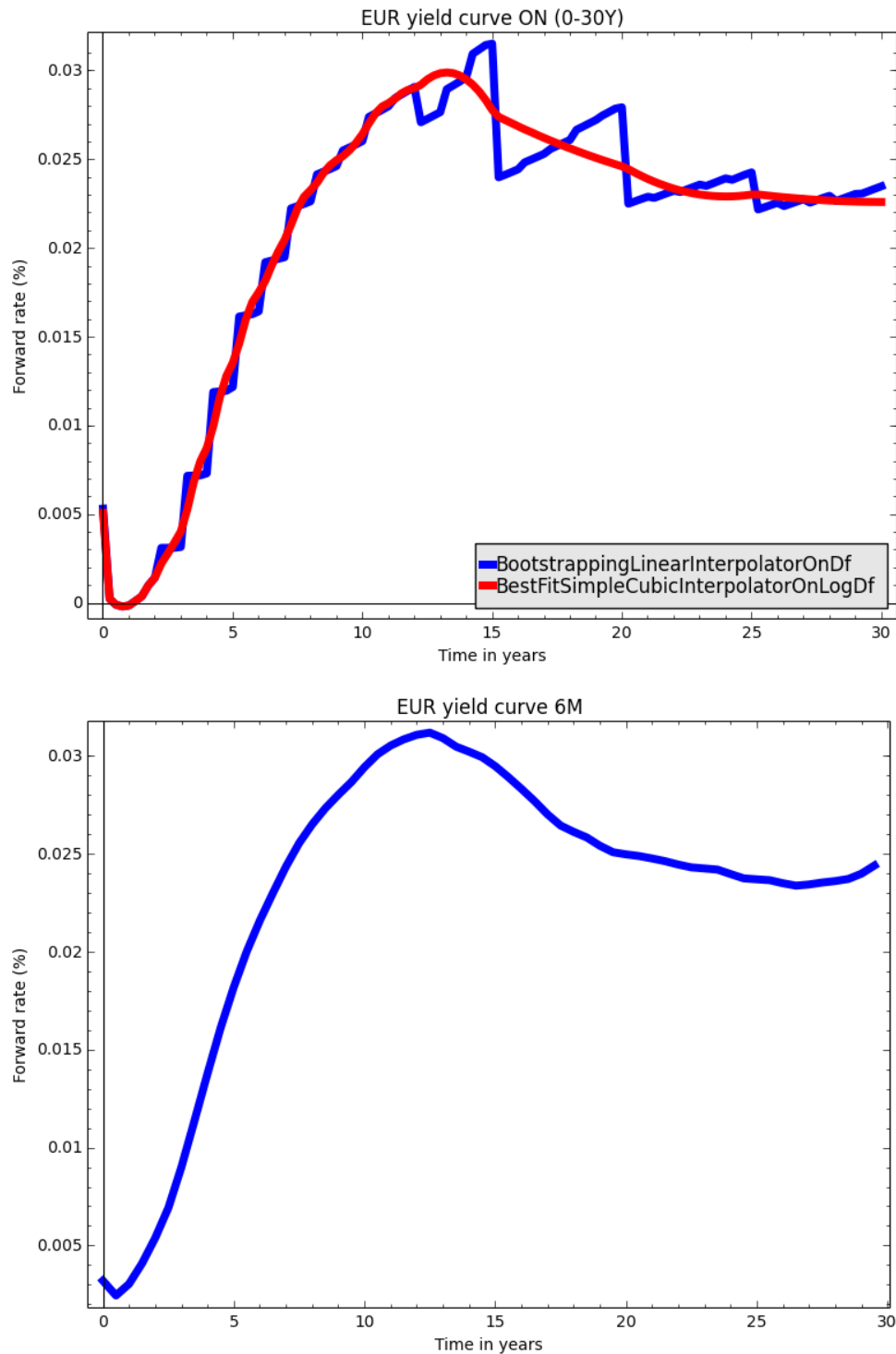


Figure 3.3: Top panel: EONIA 3M-Forward curve up to 30 years. Blue: Single Curve using Bootstrapping with Linear interpolation on log of discount factor, red: Single Curve using Best Fit with Simple Cubic interpolation on log of discount factor. Bottom panel: Euribor 6M-Forward curve up to 30 years. Multi Curve using Bootstrapping with Linear interpolation on log of discount factor.

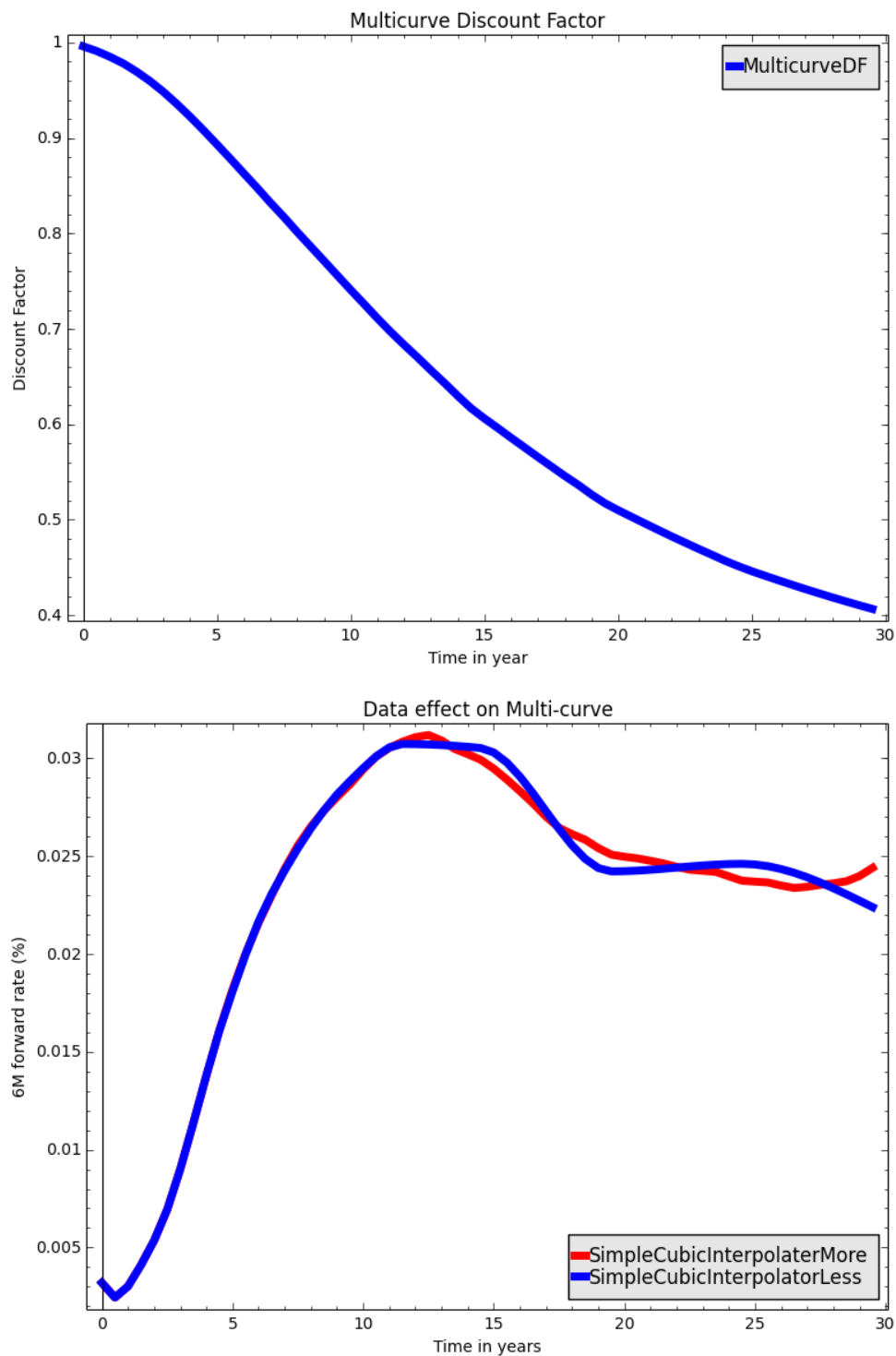


Figure 3.4: Top panel: Euribor Discount curve up to 30 years. Multi Curve discount factor. Bottom panel: effect of data on the forward curve using different interpolation. Red: Best Fit with Simple Cubic interpolation using more data, blue: Best Fit with Simple Cubic interpolation using less data.

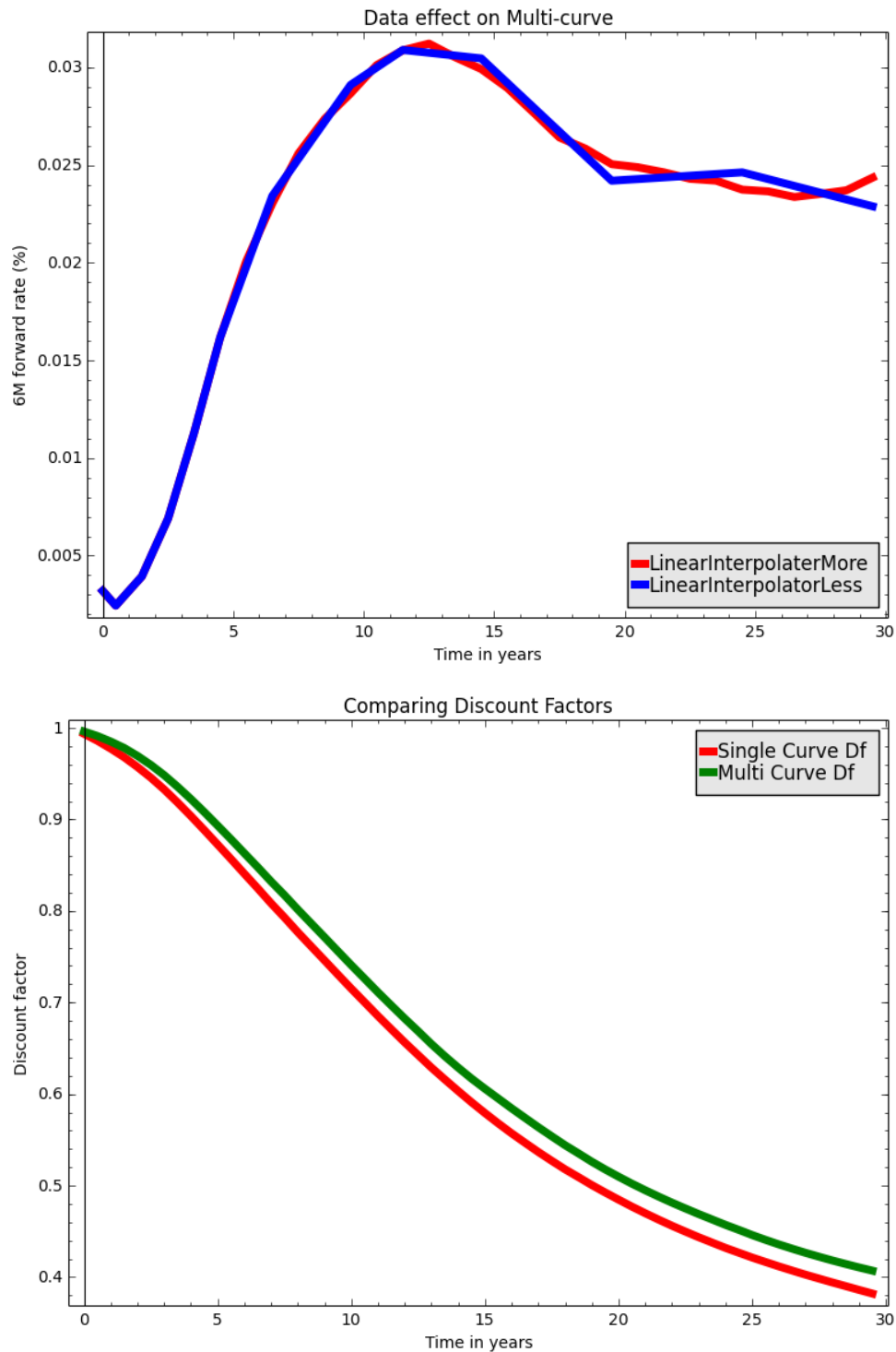


Figure 3.5: Top panel: effect of data on the forward curve using different interpolation. Red: Bootstrapping with Linear interpolation on log of discount factor using more data, blue: Bootstrapping with Linear interpolation on log of discount factor using less data. Bottom panel: comparing discount curve: Euribor vs EONIA. Red: Single Curve discount factor, green: Multi Curve discount factor.

Figure 3.3 (top panel) is the plot of the Eonia forward curve $\mathcal{C}_{\text{ON}}^F$ using the OIS market instruments shown in Table D.8 and the OIS pricing formulas discussed previously. We have “**BootstrappingLinearinterpolatorOnDf**”, where we have used the Bootstrapping methodology with Linear interpolation on discount factor. Also, we have “**BestFitSimpleCubicInterpolatorOn-LogDf**”, where we have used the Best Fit methodology with simple cubic interpolation on discount factor. The yield curve are typically used by market practitioners for different purposes. The smooth and continuous curve is normally used for pricing and hedging any collateralised financial instrument, serving as the discounting curve. The market quoted very low negative mid OIS rates for forward starting OIS on ECB dates Table D.8. Negative market rates pose new challenges to interest rate derivatives traders, interested into valuing FRA, Swaps and options including negative FRA rates and strikes. Figure 3.2 shows the short-end section of the OIS curve. Negative forward rates appear! We stress that this particular market configuration is rather exceptional, but the methodologies are robust enough to work as usual.

Figure 3.3 (bottom panel) is the plot of 6M-Forward curve up to 30 years (top panel) in the Multi Curve framework, using Bootstrapping methodology. We did not implement the best fit methodology as we did in the case of the single curve, just because the output would have led us to the same conclusion. The other yield curves: \mathcal{C}_{1M}^F , \mathcal{C}_{3M}^F and \mathcal{C}_{12M}^F can also be constructed in the same way and the observation is the same as in Figure 3.4.

Figure 3.4 (top panel) is the plot of the discount curve up to 30 years using the OIS market instruments.

Figures 3.4 (bottom panel) and 3.5 (top panel) are the plots of 6M-Forward curve. We repeat the experience as in Figures 2.4 (top panel) and 2.5, as before the density of input affects the shape of forward rates. In order to mitigate the impact from the interpolation scheme, we must build the discount curve using linear interpolation on the discount factor, and we change interpolation for forward rates.

Figure 3.5 (bottom panel) shows the comparison of discount factors calculated in multi-curve and single-curve frameworks. Clearly, we can see that discount factors calculated using OIS discounting (multi-curve) are higher than the one calculated using Euribor discounting (single-curve). The plot represents a typical and frequent situation as the OIS curve refers to something less risky than Euribor curve, so it is reasonable that discount factors are higher.

3.4 Options caps, floors and swaptions

In this section, we adapt the definitions and the formula in Section 2.5 in the multi-curve framework. We emphasis that the definition is unchanged, only some change in formulas. Particularly, we note the following change: the ATM

strike is given by

$$S(t, k, n) = \frac{\sum_{i=k}^n \delta(T_{i-1}, T_i) P_c(t, T_i) F(t; T_{i-1}, T_i)}{\sum_{i=1}^n \delta(T_{i-1}, T_i) P_c(t, T_i)}. \quad (3.4.1)$$

In the multi-curve framework, equation (2.5.6) is no longer valid, therefore the ATM strike in equation (3.4.1) above cannot be simplified. Due to the change in formula (2.5.6), it can be inferred that, there is a change as well in formulae (2.5.7) and (2.5.8) in the multi-curve framework.

3.5 Summary and conclusion

In this chapter we have given an overview of a number of changes that have taken place in the financial markets since the crash of 2007. No longer can we use single-curve models when pricing and hedging plain vanilla single-currency interest rate derivatives. Instead, we use multiple distinct curves to ensure market coherent estimation of discount factors and of forward rates with different underlying rate tenors.

We have reviewed the fundamental pricing formulas for plain vanilla interest rate derivatives, extending the single-curve framework to the modern market situation with collateral. We have applied the methodologies (Bootstrapping and Best Fit) described in Chapter 2 in the multicurve framework.

This work can be extended in the multiple-currency-multiple-curves case, in which the currency of the collateral may differ from the currency of the deal, or we consider cross currency yield curves, as discussed in (Masaaki *et al.*, 2010b), (Fujii *et al.*, 2010), (Masaaki *et al.*, 2010a), (Masaaki and Akihiko, 2011), (Castagna, 2012), (Piterbarg, 2012) and (Fujii and Takahashi, 2011).

Finally, when forward bootstrapping is not possible, due to non-local interpolation OIS-discounting, and multiple-currencies, a more general calibration approach is needed, as suggested in (Gibbs and Goyder, 2012) and (Henrard, 2013).

Chapter 4

The SABR Model

4.1 Introduction

Black-Scholes model erroneously assumed that the volatility of the underlying is constant. Therefore options on a single underlying that expire on the same date, have the same implied volatility regardless of the strikes. However, it is widely observed (Figure 4.1) that option calls with different strikes have different implied volatilities (this is known as volatility skew or the smile effect).

As it can be seen in the Figures 4.1 and 4.2, generally¹, at-the-money options tend to have lower volatilities than in- or out-of-the money options.

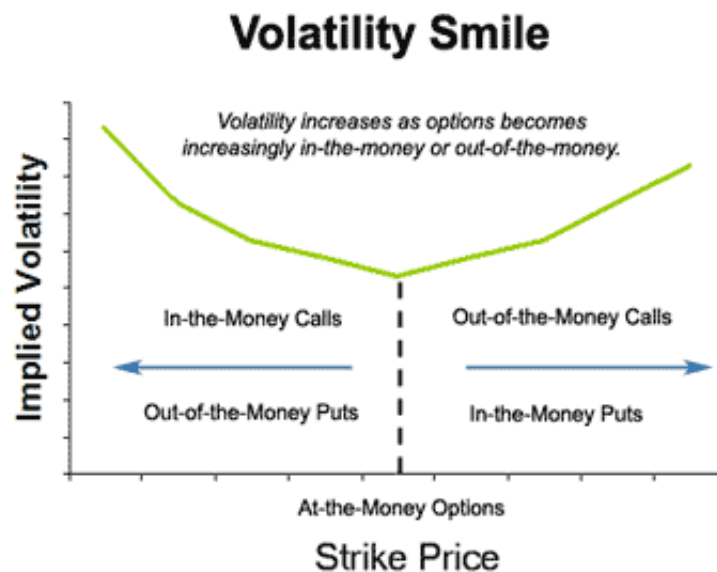


Figure 4.1: Call's volatility smile

¹We note that, other options may give different shapes.

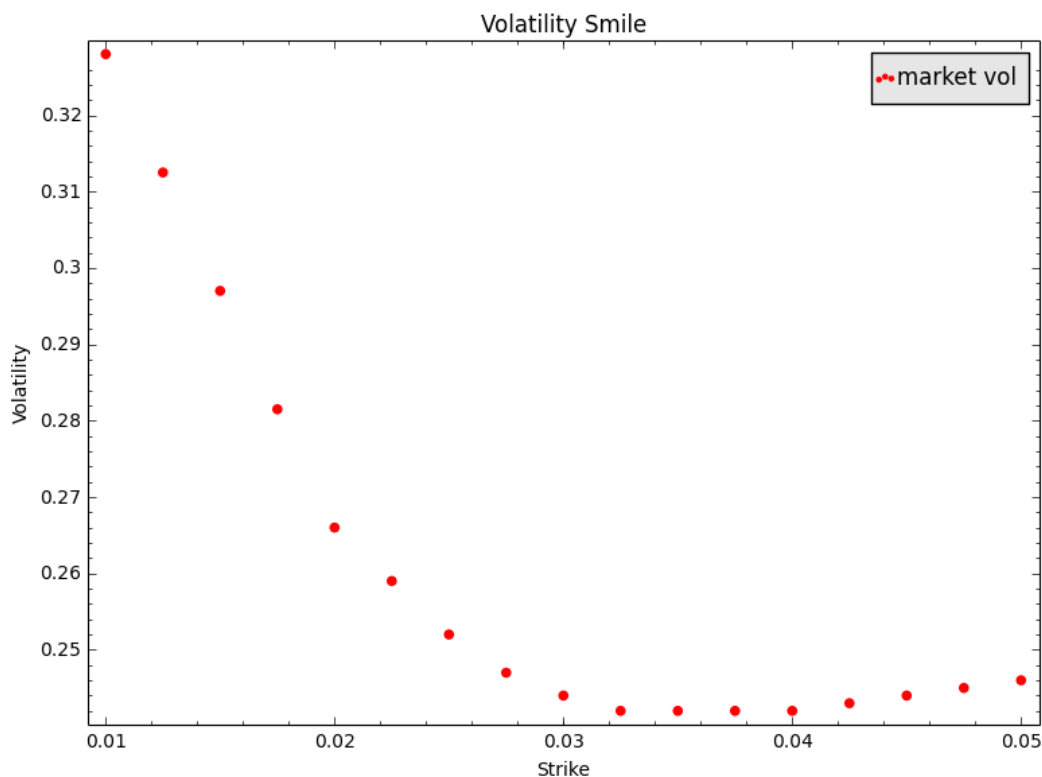


Figure 4.2: Volatility smile from data in Table D.2. Red dots: market volatility.

In order to fix the issue raised by the smile effect, that is for estimating and fitting such volatility smiles, two major approaches have been developed: local volatility and stochastic volatility.

Merton (1971) suggested to make the volatility a deterministic function of time. This would indeed explain the different volatility for different tenors, but would not explain the smile effect for different strikes.

Other local volatility models in (Dupire, 1994), Derman *et al.* (1996), (Dupire, 1997) and (Derman and Kani, 1998), suggesting that the volatility to be a deterministic function of the underlying stock and the time. However, it cannot explain the persistent smile shape which does not vanish over time with longer maturities. We can also mention local volatility models in (Berestycki *et al.*, 2002), (Chevalier, 2005) and (Gatheral *et al.*, 2012). Local volatility models are self-consistent, arbitrage-free, and can be calibrated to roughly match some observed market smiles and skews. Currently these models are the most popular way of managing smile and skew risk. Possibly they are often preferred to the stochastic volatility models for computational reasons. Nevertheless, it has recently been observed (Hagan *et al.*, 2002) that the dynamic behaviour of smiles and skews predicted by local volatility models is exactly opposite to the behaviour observed in the marketplace: local volatility

models predict that the skew moves in the opposite direction to the market level, in reality, it moves in the same direction. This leads to extremely poor hedging results within these models, and the hedges are often worse than the naive Black model hedges, because these naive hedges are in fact consistent with the smile moving in the same direction as the market.

Hence, the necessity to consider another approach: stochastic volatility. We can refer to (Hull and White, 1987), (Wiggins, 1987), (Stein and Stein, 1991), (Heston, 1993), (Hobson and Rogers, 1998), (Carr *et al.*, 2004) and (Carr and Wu, 2007). Bates (1996) introduced the first of a series of models incorporating jumps, and these were followed by (Andersen and Andreasen, 2000). Stochastic volatility models offer a widely accepted approach of incorporating into the modelling of option markets; a flexibility that accounts for the implied volatility smile or skew, see (Gatheral, 2006). One of the stochastic volatility models that has gained great popularity with practitioners is the SABR model by (Hagan *et al.*, 2005). Firstly, Hagan and Woodward (1999) used perturbation theory to find asymptotic expansions for the implied volatility of European options in a local volatility setting. Later on (Hagan *et al.*, 2002) used asymptotic methods to obtain approximations for the implied volatility in the two factor SABR models. As presented in (Hagan *et al.*, 2002), the SABR model has the advantage that it allows asset prices and market smiles to move in the same direction. Moreover, a closed-form (approximate) formula for the implied volatility is given. This implied volatility is not constant but a function of the strike price and some other model parameters. Hence the market prices and market risk, including Vanna and Volga risk, can be obtained very easily. Additionally, the SABR model provides good, and sometimes spectacular, fits to the implied volatility curves observed in the marketplace. More importantly, the SABR model captures the correct dynamics of the smile, and thus yields stable hedges.

The SABR model was generalized by (Henry-Labordere, 2005), by introducing the λ -SABR model, in which a mean-reverting drift term for the volatility is complemented. Rogers and Veraart (2008) have built an alternative model which retains many of the desirable features of the SABR model but also has exact closed-form expressions (which involve a one-dimensional integral of elementary functions) for the price of a European call option. This is not an exhaustive list of those who have contributed. Finally let us mention the work of (Henry-Labordere, 2005), they have derived a general asymptotic implied volatility at the first-order for any stochastic volatility model using the heat kernel expansion on a Riemann manifold endowed with an Abelian connection. They have obtained an asymptotic smile for a λ -SABR model corresponding to their geometric framework on the Poincaré hyperbolic plane. For $\lambda = 0$, their asymptotic implied volatility has a better approximation than the one in (Hagan *et al.*, 2002) and they have given an exact solution of the SABR model with $\beta = 0$ or 1.

We start by reviewing the SABR Model (Hagan *et al.*, 2002) and we look

at to some contributions made to improve this model.

4.2 The SABR Model: description

The SABR model is a volatility model, which attempts to capture both the correct shape of the smile, as well as the correct dynamics of the volatility smile. The model does not provide option prices exactly, instead, it gives an estimate of the implied volatility curve, which is then considered as an input in Black's model to price swaptions, caps, and other interest rate derivatives.

4.2.1 Definition

SABR is an acronym of **Stochastic, Alpha, Beta, Rho**. The SABR model was developed by Patrick Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward. Widely used by practitioners, the model is used to model forward swap rates, forward stock prices, forward Libor rates or any others forward rate. Under the T -forward measure, the SABR model is described by the following system of stochastic differential equations:

$$dF = \hat{\alpha} F^{\beta} dW_1, \quad F(0) = f \quad (4.2.1)$$

$$d\hat{\alpha} = \nu \hat{\alpha} dW_2, \quad \hat{\alpha}(0) = \alpha \quad (4.2.2)$$

$$\mathbb{E}[dW_1 dW_2] = \rho dt \quad (4.2.3)$$

where, F is any forward rate, $\hat{\alpha}$ is the volatility, and W_1 and W_2 are two correlated Wiener processes. We have the following parameters:

- α is the initial variance and satisfies the condition $\alpha \geq 0$.
- β is the exponent for the forward rate and satisfies the condition $0 \leq \beta \leq 1$.
- ν is the volatility of variance.
- ρ is the correlation between the two Wiener processes and satisfies $-1 < \rho < 1$.

We shall point out the following particular cases, when

- $\beta = 0$, the SABR model is reduced to the normal model.
- $\beta = \frac{1}{2}$, the SABR model is reduced to the stochastic Cox–Ingersoll–Ross model.
- $\beta = 1$, the SABR model is reduced to the stochastic log-normal model.
- $\nu = 0$, the SABR model is reduced to the CEV (Constant Elasticity of Variance) model.

4.2.2 SABR implied volatility

The Black's formula for the price of a European call option is given by

$$C(F, K, \sigma_B, t_{ex}) = e^{-rt_{ex}} [F\mathcal{N}(d_+) - K\mathcal{N}(d_-)] \quad (4.2.4)$$

where t_{ex} is the time to maturity, K is the strike price, F is the forward of a log-normal underlying with constant risk-free interest rate r and with constant volatility σ_B . Here

$$d_{\pm} = \frac{\ln(\frac{F}{K}) \pm \frac{1}{2}\sigma_B^2 t_{ex}}{\sigma_B \sqrt{t_{ex}}}$$

and

$$\mathcal{N}(t) = \frac{1}{2\pi} \int_{-\infty}^t e^{-\frac{x^2}{2}} dx.$$

The value of a European option call on the forward F struck at K , which expires t_{ex} years from now is given by:

$$C_t = P \times \mathbb{E} [\max(F_{t_{ex}} - K, 0)] \quad (4.2.5)$$

where P is the corresponding discount factor and the expectation is calculated under the probability distribution of the process F_t .

No closed form expression for this probability distribution is known, only for the special cases: $\beta = 0$ and $\beta = 1$. Asymptotic expansion in the parameter $\varepsilon = t_{ex}\alpha^2$ is used to solve the general case. This parameter is small under market condition and the approximate solution has the following benefits: quite accurate, simple functional form, very easy to implement in computer code, good enough to handle risk management of large portfolios of options in real time.

The prices of European call options in the SABR model are forced into the form of the Black model valuation formula (equation 4.2.4). The implied volatility $\sigma_B(f, K)$ is approximately given by:

$$\sigma_B(f, K) \simeq U \left(\frac{z}{X(z)} \right) V \quad (4.2.6)$$

where

$$U = \alpha \left\{ (fK)^{(1-\beta)/2} \left[1 + \frac{(1-\beta)^2}{24} \log^2 \left(\frac{f}{K} \right) + \frac{(1-\beta)^4}{1920} \log^4 \left(\frac{f}{K} \right) \right] \right\}^{-1}, \quad (4.2.7)$$

$$V = \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{\rho\beta\nu\alpha}{4(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} \nu^2 \right] t_{ex} + \dots \right\}. \quad (4.2.8)$$

Here

$$z = \frac{\nu}{\alpha} \left(fK \right)^{(1-\beta)/2} \log \left(\frac{f}{K} \right) \quad (4.2.9)$$

and $X(z)$ is defined by

$$X(z) = \log \left\{ \frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho} \right\}. \quad (4.2.10)$$

The proof of this result can be found in Appendix F. We note the particular case of at-the-money options, options on struck at $K = f$; formula (4.2.6)-(4.2.10) reduces to

$$\sigma_{\text{ATM}} = \frac{\alpha}{f^{(1-\beta)}} \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho\beta\nu\alpha}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24} \nu^2 \right] t_{ex} + \dots \right\}. \quad (4.2.11)$$

We note that, when $K = f$, z in equation (4.2.9) $\rightarrow 0$ and X in equation (4.2.10) $\rightarrow 0$. Using L'Hôpital's rule, we found that $z/X \rightarrow 1$; hence the above result. We discuss the qualitative behaviour of formula (4.2.6) and its use in managing smile in the next sections.

4.3 Model dynamics

In this section, we study the qualitative behaviour of the SABR parameters (this is crucial, because it helps to infer the behaviour of the smile), as well as methods of calibration. Because of the widely separated roles these parameters play, the fitted parameter values tend to be very stable, even in the presence of large amounts of market noise.

4.3.1 Qualitative behaviour

To facilitate the comprehension of the qualitative behaviour of the model, formula (4.2.6) is approximated as:

$$\begin{aligned} \sigma_B(f, K) = \frac{\alpha}{f^{(1-\beta)}} & \left\{ 1 - \frac{1}{2} (1 - \beta - \rho\lambda) \log \left(\frac{f}{K} \right) \right. \\ & \left. + \frac{1}{12} \left[(1 - \beta)^2 + (2 - 3\rho^2)\lambda^2 \right] \log^2 \left(\frac{f}{K} \right) + \dots \right\} \end{aligned} \quad (4.3.1)$$

with the strike K not too far from the current forward f .

Compared to the quantity $\alpha f^{\beta-1}$, the quantity λ which is defined by

$$\lambda = \frac{\nu}{\alpha} f^{1-\beta} \quad (4.3.2)$$

is a good measure of the strength of the volatility of volatility ("volvol") ν .

- The curve of the implied volatility $\sigma_B(f, K)$ as a function of K for a fixed f is known as the **smile** or **skew** whereas the curve of the ATM volatility $\sigma_{\text{ATM}} = \sigma_B(f, f)$ is known as the **backbone**. Note that, using equation (4.3.1), $\sigma_{\text{ATM}} = \sigma_B(f, f) = \alpha/f^{1-\beta}$ and it can be inferred that when $\beta = 0$ we observe a steeply downward sloping backbone and for $\beta = 1$ we observe a flat backbone.
- The quantity

$$-\frac{1}{2}(1 - \beta - \rho\lambda) \log\left(\frac{f}{K}\right)$$

in equation (4.3.1), which is function of K , represents the skew or the slope of the implied volatility. It can be viewed separately as a summation of two terms. The first term

$$-\frac{1}{2}(1 - \beta) \log\left(\frac{f}{K}\right)$$

is known as the beta skew, since $0 \leq \beta \leq 1$ implies that

$$-\frac{1}{2} \leq -\frac{1}{2}(1 - \beta) \leq 0,$$

hence this term is downward sloping for $K > f$. The second term

$$\frac{1}{2}\rho\lambda \log\left(\frac{f}{K}\right)$$

represents the **vanna** skew.

- Likewise, the last quantity

$$\left[(1 - \beta)^2 + (2 - 3\rho^2)\lambda^2\right] \log^2\left(\frac{f}{K}\right)$$

in equation (4.3.1) can be viewed separately as a summation of two terms. The first term

$$(1 - \beta)^2 \log^2\left(\frac{f}{K}\right)$$

is a smile (quadratic) term. This term is overshadowed by the beta skew and is less influential on this skew. The second term

$$(2 - 3\rho^2)\lambda^2 \log^2\left(\frac{f}{K}\right)$$

is the smile induced by the **volga** (volatility-gamma) effect.

4.3.2 Parameter estimation

At a fixed f , the implied volatility σ_B is reduced to a function of K when the parameters α, β, ρ and ν are estimated. Here, we examine how these parameters are estimated.

The exponent β and correlation ρ both cause a downward sloping skew in $\sigma_B(K, f)$ as the strike K varies. Usually, the parameter β is estimated first and from experience it is observed that market smiles can be fitted equally well with any specific value of β . Once β is known, there are two ways of estimating α, ρ and ν : we can calibrate α, ρ and ν directly (in one step) or we can calibrate ρ and ν directly (first step) and deduce α from ρ, ν and σ_{ATM} (second step).

1. Estimating β

On the one hand, historical observations of the backbone can be used to estimate β . By taking the log side by side in equation (4.2.11), we get

$$\begin{aligned} \log \sigma_{\text{ATM}} = & \log \alpha - (1 - \beta) \log f + \\ & \log \left\{ 1 + \left[\frac{(1 - \beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho \beta \nu \alpha}{f^{(1-\beta)}} + \frac{2 - 3\rho^2}{24} \nu^2 \right] t_{\text{ex}} + \dots \right\}. \end{aligned} \quad (4.3.3)$$

As the term

$$\left[\frac{(1 - \beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho \beta \nu \alpha}{f^{(1-\beta)}} + \frac{2 - 3\rho^2}{24} \nu^2 \right] t_{\text{ex}}$$

is generally less than one or two per cent, it is usually ignored; therefore equation (4.3.3) is approximated as

$$\log \sigma_{\text{ATM}} \simeq \log \alpha - (1 - \beta) \log f. \quad (4.3.4)$$

On the other hand, β can be estimated from what is called “aesthetic considerations”: $\beta = 0$ (stochastic normal), $\beta = 0.5$ (stochastic CIR) or $\beta = 1$ (stochastic lognormal) models. Supporters of the choice $\beta = 0$ believe that a normal model is a powerful tool for managing risks and would postulate that, for markets where $f < 0$ or $f \simeq 0$, $\beta = 0$. Supporters of the choice $\beta = 0.5$ are usually US interest rate desks that have developed trust in CIR models. Finally, supporters of the choice $\beta = 1$ include desks trading foreign exchange options, they believe that log normal models are “more natural” or believe that the horizontal backbone best represents their market.

It is more appropriate in practice to use the ATM volatility $\sigma_{\text{ATM}}, \beta, \rho$ and ν as the SABR parameters instead of the original parameters α, β, ρ, ν .

2. Estimating α, ρ and ν

As we mentioned earlier, once β is known, there are two ways of estimating α, ρ and ν . The first method consists of minimizing the errors between the market volatilities σ_j^{mkt} and the model volatilities σ_B for the same maturity t_{ex} . That is

$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \operatorname{argmin}_{\alpha, \rho, \nu} \sum_j \left[\sigma_B(f_j, K_j; \alpha, \rho, \nu) - \sigma_j^{mkt} \right]^2. \quad (4.3.5)$$

The second method consists of estimating directly ρ and ν as above; that is by minimizing the errors between the market volatilities σ_j^{mkt} and the model volatilities σ_B for the same maturity t_{ex}

$$(\hat{\rho}, \hat{\nu}) = \operatorname{argmin}_{\rho, \nu} \sum_j \left[\sigma_B(f_j, K_j; \alpha, \rho, \nu) - \sigma_j^{mkt} \right]^2. \quad (4.3.6)$$

Once we know ρ and ν , we can use equation (4.2.11) to get α . We note that, α is the root of the cubic equation

$$\sigma_{\text{ATM}} f^{(1-\beta)} - \alpha \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho\beta\nu\alpha}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24} \nu^2 \right] t_{ex} \right\} = 0,$$

nothing else than

$$\frac{(1-\beta)^2 t_{ex}}{24 f^{2-2\beta}} \alpha^3 + \frac{\rho\beta\nu t_{ex}}{4 f^{(1-\beta)}} \alpha^2 + \left(1 + \frac{2-3\rho^2}{24} \nu^2 t_{ex} \right) \alpha - \sigma_{\text{ATM}} f^{(1-\beta)} = 0. \quad (4.3.7)$$

The selection of the smallest positive root of this last equation is suggested.

When applying SABR to options on US dollar interest rate (Eurodollar future options, caps/floors and European swaptions), some observations were made in (Hagan *et al.*, 2002). First, the smile and skew depend heavily on the time-to-exercise for Eurodollar future options and swaptions. The smile is pronounced for short-dated options and flattens for longer dated options; the skew is overwhelmed by the smile for short-dated options, but is important for long-dated options. Second, in most markets there is a strong smile for short-dated options which relaxes as the time-to-expiry increases; consequently the volatility of volatility ν is large for short dated options and smaller for long-dated options, regardless of the particular underlying. Finally in some markets a nearly flat skew for short maturity options develops into a strongly downward sloping skew for longer maturities. In other markets there is a strong downward skew for all option maturities, and in still other markets the skew is close to zero for all maturities.

4.3.3 Dynamics of the parameters

The three parameters α, ρ and ν have different effects on the curve. We will adjust each parameter while keeping the others constant and observe the result in the smile's shape.

4.3.3.1 Dynamics of Beta

Figure 4.3 shows the SABR volatility smile for a swaption calibrated using Method 1 (we refer to Section 4.3.2 for more details) for $\beta = 0.5$. Resulting parameters are $\rho = 0.084$, $\nu = 0.674$ and $\alpha = 0.058$. The sse (sum squared error) for the fit when $\rho = 0.084$ is approximately 0.015. Afterwards, we consider the cases $\beta = 0.7$ and $\beta = 0.3$ while keeping the rest of the parameters constant. From figure, it can be seen that a changed in beta, does not fit to the market volatility to the calibration. Moreover, there is a rather big effect on the curvature of the smile for a change in beta where the left hand side of ATM point is more effected than the right hand side.

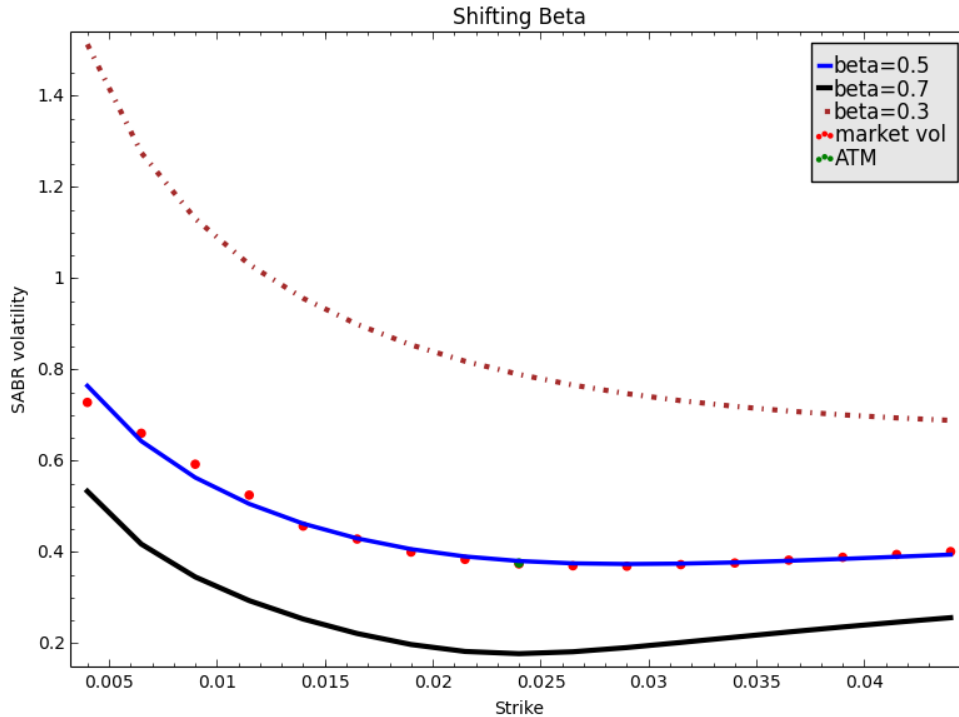


Figure 4.3: Dynamics of the parameter β . Blue: SABR volatility smile for a swaption calibrated using Method 1 for $\beta = 0.5$. Resulting parameters are $\rho = 0.084$, $\nu = 0.674$ and $\alpha = 0.058$. Brown: SABR volatility smile for $\beta = 0.7$ and black: SABR volatility smile for $\beta = 0.3$, red: market volatilities, green: at the money.

4.3.3.2 Dynamics of Rho

Figure 4.4 shows the SABR volatility smile for a swaption calibrated using Method 1 (we refer to Section 4.3.2 for more details) for $\beta = 0.5$. Resulting parameters are $\rho = 0.084$, $\nu = 0.674$ and $\alpha = 0.058$. The **sse** for the fit when $\rho = 0.084$ is approximately 0.015. Afterwards, we consider the cases $\rho = 0.334$ and $\rho = -0.166$ while keeping the rest of the parameters constant. One can see that an increased ρ rotates the curve counter clockwise, creating a flatter smile. On the contrary, a decrease ρ would lead to a clockwise rotation of the SABR curve and hence a steeper smile. This is exactly as it has been claimed in (Hagan *et al.*, 2002) that rho controls the skew of the curve.

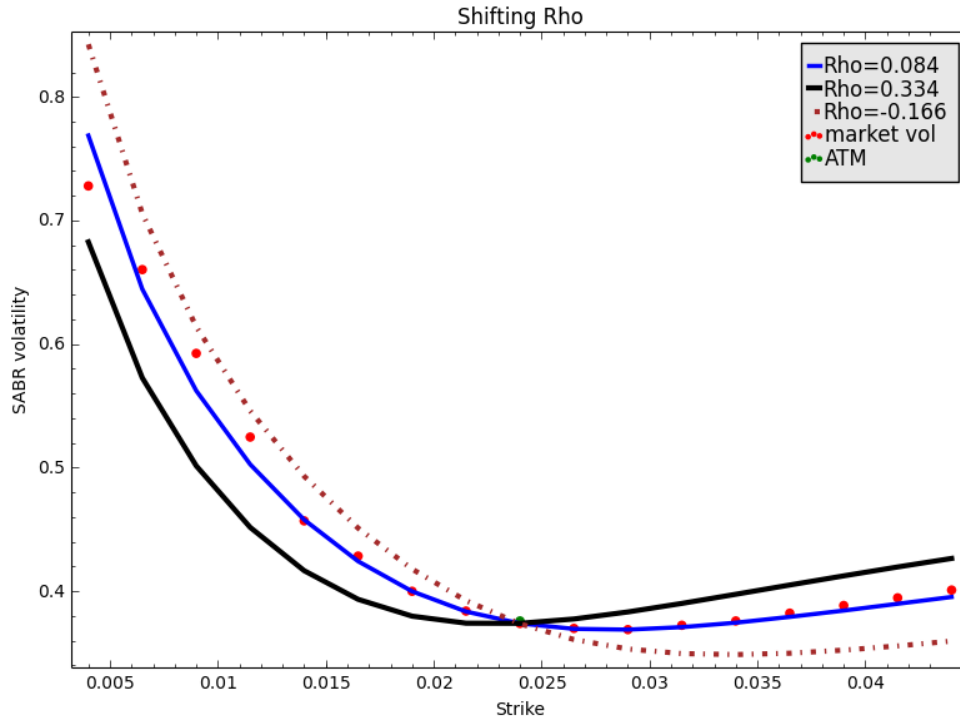


Figure 4.4: Dynamics of the parameter ρ . Blue: SABR volatility smile for a swaption calibrated using Method 1 for $\rho = 0.084$. Resulting parameters are $\beta = 0.5$, $\nu = 0.674$ and $\alpha = 0.058$. Brown: SABR volatility smile for $\rho = -0.166$ and black: SABR volatility smile for $\rho = 0.334$, red: market volatilities, green: at the money.

4.3.3.3 Dynamics of Nu

Parameter ν or volvol is the volatility of the volatility. This parameter should be directly calibrated using market data with the constraint that $\nu > 0$.

Figure 4.5 shows the SABR volatility smile for a swaption calibrated using Method 1 (we refer to Section 4.3.2 for more details) for $\beta = 0.5$. Resulting parameters are $\rho = 0.084$, $\nu = 0.674$ and $\alpha = 0.058$. The **sse** for the fit when $\nu = 0.674$ is approximately 0.015. Afterwards, we consider the cases $\nu = 1.174$ and $\nu = 0.174$ while keeping the rest of the parameters constant. After calibration using predetermined $\beta = 0.5$, we get $\nu = 0.674$. With an increase and decrease in ν by 0.5, we obtain a more convex and a flatter smile around ATM point respectively. One can see, that the volvol controls how much smile the curve exhibits, where an increase in ν would increase the smile effect of the curve.

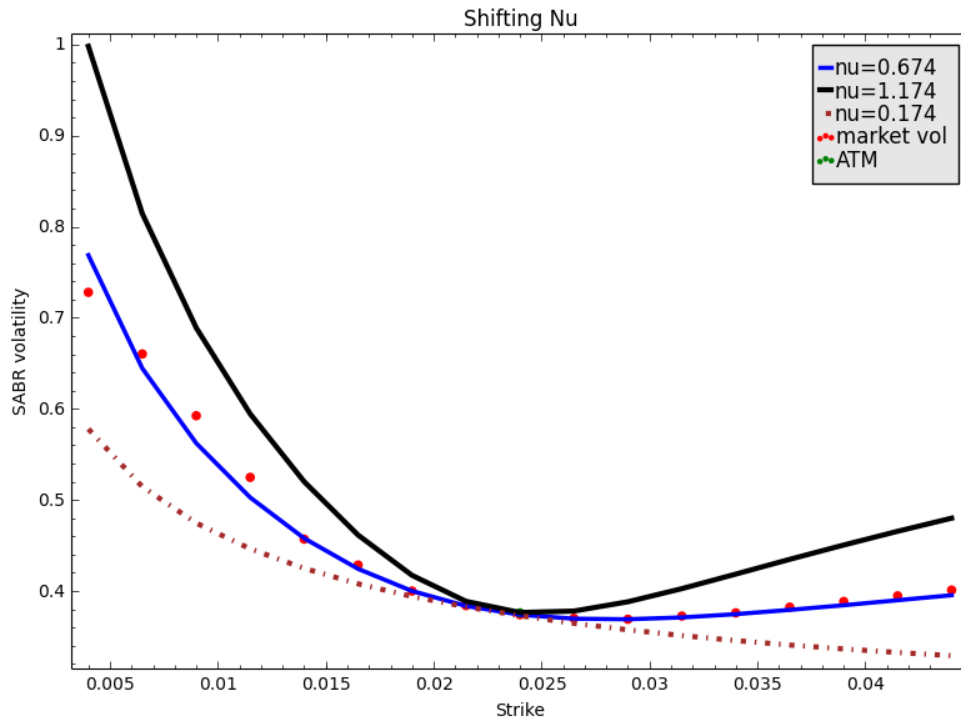


Figure 4.5: Dynamics of the parameter ν . Blue: SABR volatility smile for a swaption calibrated using Method 1 for $\nu = 0.674$. Resulting parameters are $\rho = 0.084$, $\beta = 0.5$ and $\alpha = 0.058$. Brown: SABR volatility smile for $\nu = 0.174$ and black: SABR volatility smile for $\nu = 1.174$, red: market volatilities, green: at the money.

4.3.3.4 Dynamics of Alpha

Parameter α is different to parameters β , ν and ρ since it is a stochastic parameter. An increase in this parameter will lead to an upward shift of the entire smile while a decrease will result in a downward shift. This observation can

be made after setting α to be the initial volatility where the entire stochastic process would begin from. As a result, α should govern the vertical location of the smile rather than the smile's shape. Also for this reason $\alpha > 0$ since we can not experience non-positive volatility.

Figure 4.6 shows the SABR volatility smile for a swaption calibrated using Method 1 (we refer to Section 4.3.2 for more details) for $\beta = 0.5$. Resulting parameters are $\rho = 0.084$, $\nu = 0.674$ and $\alpha = 0.058$. The sse for the fit when $\alpha = 0.058$ is approximately 0.015. Afterwards, we consider the cases $\alpha = 0.068$ and $\alpha = 0.048$ while keeping the rest of the parameters constant. In Figure 4.6, we can see that the curvature of the SABR curves seem to remain constant when alpha is increased and decreased.

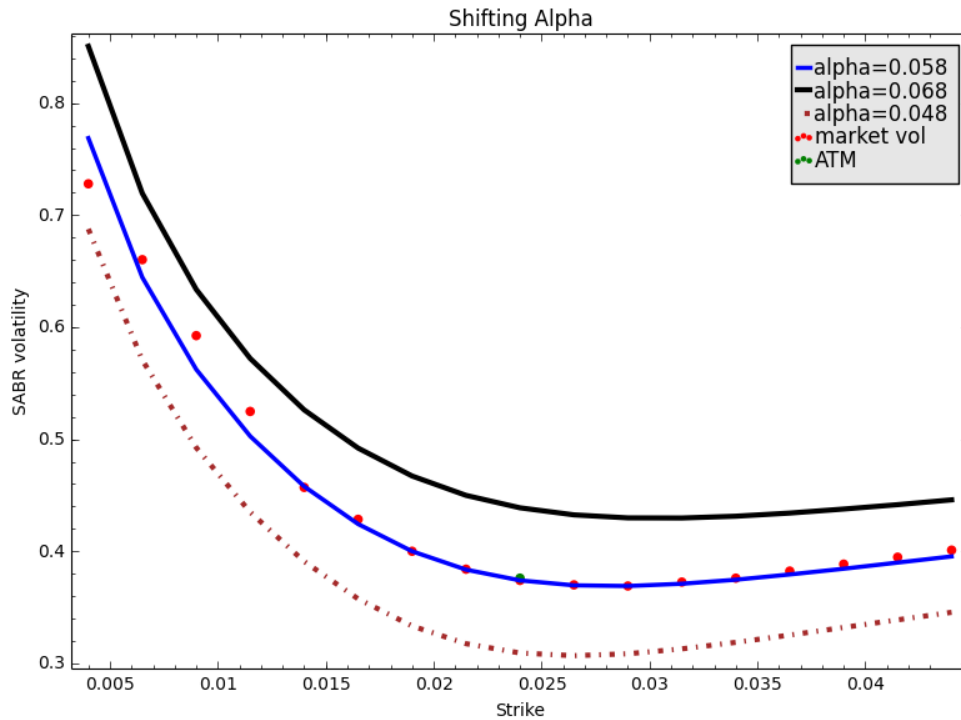


Figure 4.6: Dynamics of the parameter α . Blue: SABR volatility smile for a swaption calibrated using Method 1 for $\alpha = 0.058$. Resulting parameters are $\rho = 0.084$, $\nu = 0.674$ and $\beta = 0.5$. Brown: SABR volatility smile for $\alpha = 0.048$ and black: SABR volatility smile for $\alpha = 0.068$, red: market volatilities, green: at the money.

4.3.3.5 Dynamics of f

Today's forward price is denoted by f . Figure 4.7 shows the behaviour of the SABR volatility smile for a swaption when shifting the value of f . We consider

all the parameters β , ρ , ν and α as constant. Then, from a given initial f , we shift the value of f by adding successively -80bps and 80bps. Figure 4.7 shows how much the smile moves by changing the value of f . This is very important to note as we will see in Chapter 5.

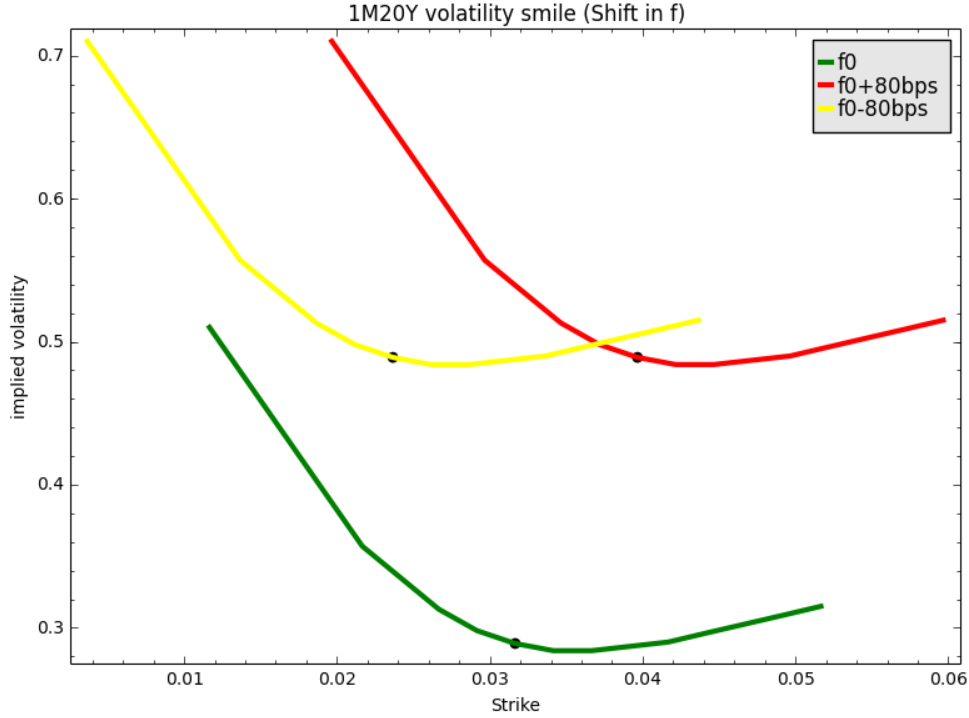


Figure 4.7: Volatility smile shifting f . Green: SABR volatility smile for a swaption for $f = f_0$. All parameters are kept constant $\rho = 0.084$, $\nu = 0.674$ and $\alpha = 0.058$. Yellow: SABR volatility smile for $f = f_0 - 80$ and red: SABR volatility smile for $f = f_0 + 80$.

4.3.4 Backbone

In (Hagan *et al.*, 2002), the backbone is defined as the curve of the ATM volatility $\sigma_{\text{ATM}} = \sigma_B(f, f)$. Note that, using equation (3.3.1), we have

$$\sigma_B(f, f) = \alpha / f^{1-\beta}.$$

The backbone is observed to be dependent almost entirely on the parameter β used as the exponent of the price process. In the first instance, we calibrate the model with $\beta = 0$ for a swaption with ATM rate of 2.4%. Afterwards, we consider the cases $f = 2.9\%$ and $f = 1.9\%$. From Figure 4.8, we can observe that the ATM volatility move accordingly.

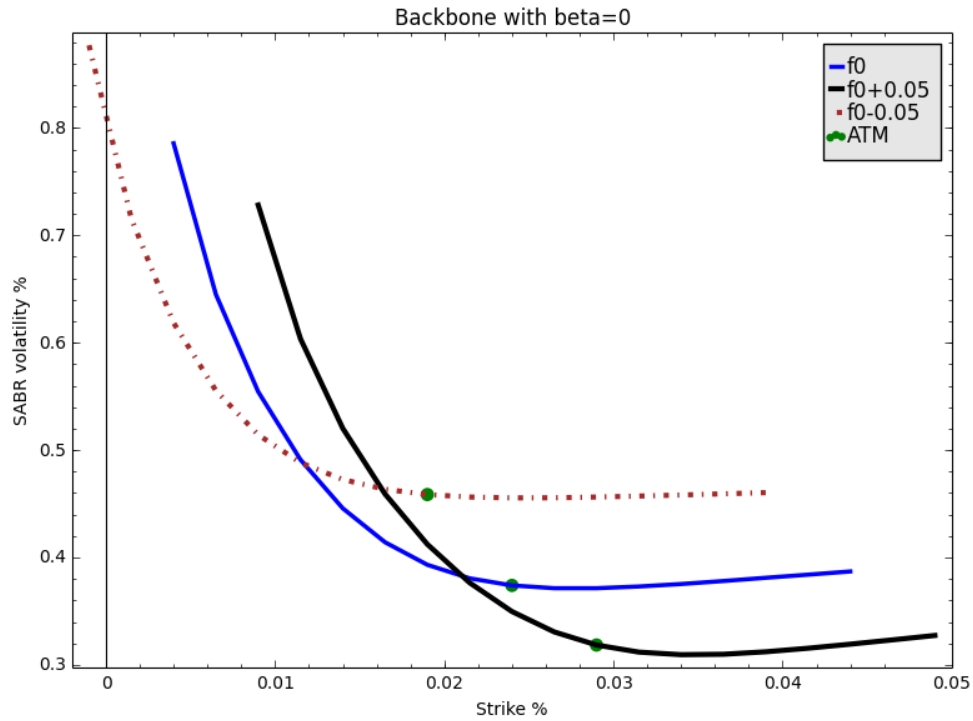


Figure 4.8: Shifting f in the backbone for $\beta = 0$. Blue: SABR volatility smile for a swaption for $f = f_0$. All parameters are kept constant. Brown: SABR volatility smile for $f = f_0 - 0.05$ and black: SABR volatility smile for $f = f_0 + 0.05$.

In the second instance, instead we calibrate the model with $\beta = 1$ and then change the forward rate, the result is different. From Figure 4.9, we can see that increasing and decreasing f only move the curve on the horizontal axis.

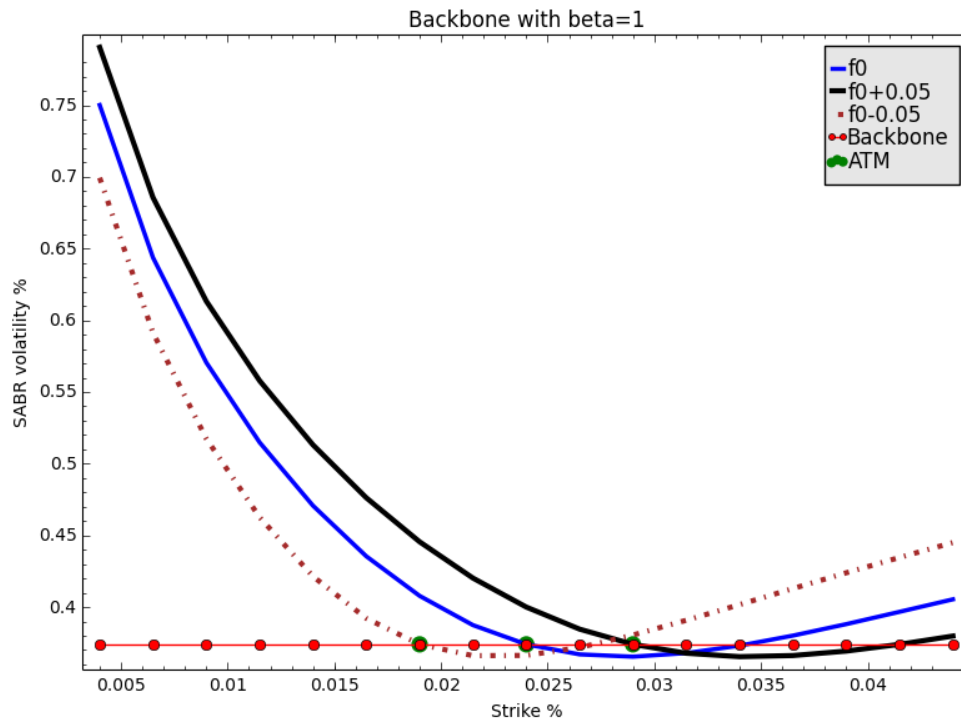


Figure 4.9: Shifting f in the backbone for $\beta = 1$. Blue: SABR volatility smile for a swaption for $f = f_0$. All parameters are kept constant. Brown: SABR volatility smile for $f = f_0 - 0.05$ and black: SABR volatility smile for $f = f_0 + 0.05$.

SABR drawbacks

Though, the model is very popular nowadays, the SABR model (Hagan *et al.*, 2002) is based on approximation techniques and it turns out; first that the option pricing formula can be inaccurate, in particular for high volatilities, low strikes and long expiries. Second the asset distribution it implies is not guaranteed to be arbitrage-free, i.e. the formula can result in a negative probability density function in some regions, when applied to long-dated options. Third, usually overlooked, the sensitivities to SABR parameters may not be adequate tools for risk management, since in this case SABR parameters can cease to be financially meaningful quantities.

There is a growing of consideration in the literature around the SABR model (Hagan *et al.*, 2002), many papers have proposed some methodologies for avoiding these issues as well as new approaches for dealing with the model. Let us mention some: (Lee, 2004), (Benaim *et al.*, 2008), (Oblój, 2008), (Hagan *et al.*, 2014), (Morini and Mercurio, 2007), (West, 2005), (Roper, 2010), (Doust, 2012) among others.

This inconsistency has also affected on the risks calculated from this model. Delta and Vega hedging fall due to the wrong dynamics predicted.

4.4 Refinement of the SABR model

The Taylor expansion of the implied volatility surface $I(x, \tau)$, at maturity τ , is given by

$$I(x, \tau) = I^0(1 + I^1(x)\tau) + \mathcal{O}(\tau^2). \quad (4.4.1)$$

The first explicit expressions for I^0 and I^1 were obtained in (Hagan *et al.*, 2002). They used perturbation theory combined with impressive intuition (we refer to Appendix F for more details). More recently, Berestycki *et al.* (2004) treated the subject in a rigorous analytical manner. They proved in particular that $I^0(x) = \lim_{\tau \rightarrow 0} I(x, \tau)$ is well defined.

When it comes to the formula of $I^0(x)$ with $\beta < 1$, Oblój (2008) has suggested to use the formula of (Berestycki *et al.*, 2004) for the following arguments. First, theoretically, the formula of $I^0(x)$ in (Hagan *et al.*, 2002) is inconsistent as $\beta \rightarrow 1$. Considering the last formula in the table below (Table 4.1), there is no denying that $I^0(x)$ in (Hagan *et al.*, 2002) does not converge to the known value for $I^0(x)|_{\beta=1}$ as $\beta \rightarrow 1$. However, there is no denying that $I^0(x)$ in (Berestycki *et al.*, 2004) does converge to the known value for $I^0(x)|_{\beta=1}$ as $\beta \rightarrow 1$. Second, the formula of (Hagan *et al.*, 2002) is known to produce wrong prices in region of small strikes for large maturities. It assigns a negative price to a structure with a positive payoff, or equivalently it implies a negative probability density for the stock price process in some region. If one uses the formula we derive here the problem either appears in yet lower strikes or disappears completely.

Let $I_H^0(x)$ and $I_B^0(x)$ be the formulae for I^0 from (Hagan *et al.*, 2002) and (Berestycki *et al.*, 2004) respectively. The implied volatility $I_H(x)$ is given by

$$I_H(x) = I_H^0(1 + I_H^1(x)\tau), \quad (4.4.2)$$

where

$$I_H^1(x) = \frac{(1-\beta)^2}{24} \frac{\alpha^2}{(sK)^{1-\beta}} + \frac{1}{4} \frac{\rho\nu\alpha\beta}{(sK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} \nu^2.$$

In (Oblój, 2008) the implied volatility is defined by

$$I_B(x) = I_B^0(1 + I_H^1(x)\tau).$$

| I^0 | (Hagan <i>et al.</i> , 2002) | (Berestycki <i>et al.</i> , 2004) | Notation |
|---------------------|--|---|---|
| $I^0(0)$ | $\alpha K^{\beta-1}$ | $\alpha K^{\beta-1}$ | |
| $I^0(x) _{\nu=0}$ | $\frac{x\alpha(1-\beta)}{s^{1-\beta}-K^{1-\beta}}$ | $\frac{x\alpha(1-\beta)}{s^{1-\beta}-K^{1-\beta}}$ | |
| $I^0(x) _{\beta=1}$ | $\nu x / \log\left(\frac{\sqrt{1-2\rho z+z^2+z-\rho}}{1-\rho}\right)$ | $\nu x / \log\left(\frac{\sqrt{1-2\rho z+z^2+z-\rho}}{1-\rho}\right)$ | $z = \frac{\nu x}{\alpha}$ |
| $I^0(x) _{\beta<1}$ | $\nu x \frac{\zeta}{z} / \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2+\zeta-\rho}}{1-\rho}\right)$ | $\nu x / \log\left(\frac{\sqrt{1-2\rho z+z^2+z-\rho}}{1-\rho}\right)$ | $z = \frac{\nu}{\alpha} \frac{s^{1-\beta}-K^{1-\beta}}{1-\beta}$ $\zeta = \frac{\nu}{\alpha} \frac{s-K}{(sK)^{\beta/2}}$ |

Table 4.1: Comparison of I^0 term Hagan vs Berestycki

4.5 Calibration of the SABR model

In this section, we present different calibrations of the SABR model with Matlab² and results. We use the volatility data in appendix D.2 and the methodologies presented in Sections 4.3.2 and 4.4.

In the first instance, we fix different values of beta: 0, 0.5 and 1. Afterwards, we calibrate the rest of the parameters: ρ , ν and α . In the second instance, we estimate all parameter values using only the given market smile. Parameters that minimize our **sse** (sum squared error) are found by calling the function “**fminsearch**” in Matlab.

4.5.1 Different values of beta

Considering only methods described in Section 4.3.2 (i.e. without refinements), we calibrate the SABR model for different values of beta: 0, 0.5, 1. All the results are summarized in Tables 4.2 and 4.3. In both methods, we observe an insignificant difference in term of error for all betas. We observe the smallest error under the second calibration for $\beta = 1$, however, for $\beta = 0$, we have the largest error compared to other values of beta. Figures 4.10 and 4.11 illustrate the fit to market data.

| beta | rho | volvol | alpha | error |
|------|----------|---------|-----------|-----------|
| 0 | 0.64 | 0.31459 | 0.0071249 | 0.011801 |
| 0.5 | 0.22619 | 0.2964 | 0.040393 | 0.0062754 |
| 1 | -0.19683 | 0.33847 | 0.23802 | 0.0050049 |

Table 4.2: Method 1 estimated for different beta

²We use functions written by of **Fabrice Douglas Rouah** (available at <http://www.volopta.com>).

| beta | rho | volvol | alpha | error |
|------|----------|---------|-----------|-----------|
| 0 | 0.68486 | 0.28987 | 0.0072634 | 0.011431 |
| 0.5 | 0.22995 | 0.28271 | 0.04082 | 0.0064435 |
| 1 | -0.20166 | 0.33447 | 0.23894 | 0.0051533 |

Table 4.3: Method 2 estimated for different beta

In Figure 4.10, we calibrate a 5Y12Y swaption by using Method 1 (top panel) and Method 2 (bottom panel). The estimated SABR parameters can be found in Tables 4.2 and 4.3. It can be seen from Figure 4.10, there is a point of intersection (very close to the ATM point of the swaption) for the three SABR curves. The reason been that we only minimize the sum of the squared errors for two parameters (i.e. ν and ρ). Similar results were observed with 1M5Y swaption (Figure 4.11), though the ATM point of the swaption is no longer the point of intersection for the three SABR curves (except when $\beta = 0.5$).

We can see that both methods produce spectacular fit to the market data. The chosen method should be based on whether we want to fit the SABR curve through the point of ATM volatility. The calibration produce a good fit for all the betas. We can join (Hagan *et al.*, 2002) in claiming that the beta value does not have such a great impact of the quality of the fit.

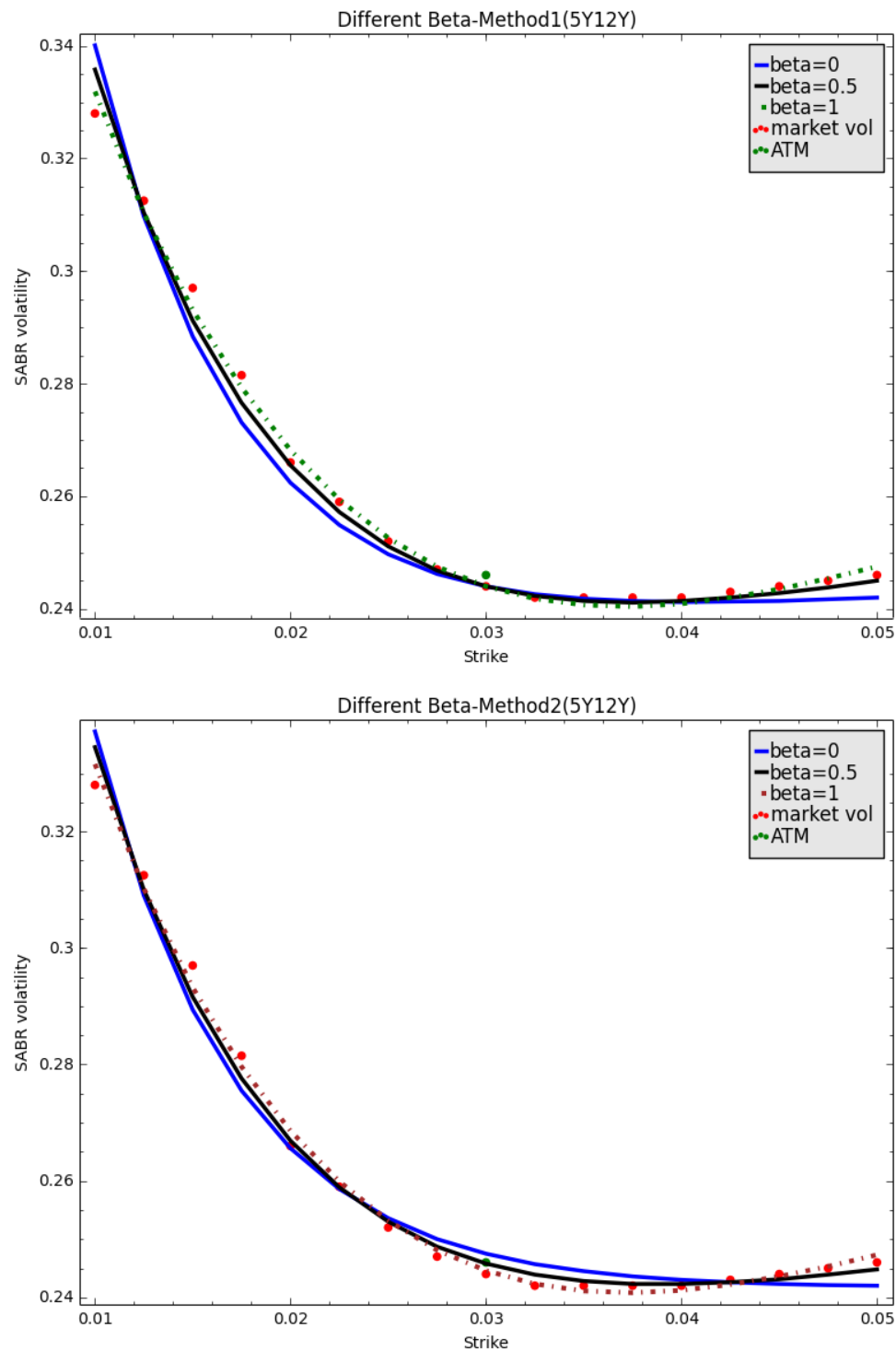


Figure 4.10: 5Y12Y Calibration with different beta using Method 1 (top panel), 5Y12Y Calibration with different beta using Method 2 (bottom panel). Blue: SABR volatility smile for $\beta = 0$, black: SABR volatility smile for $\beta = 0.5$, brown: SABR volatility smile for $\beta = 1$, red: market volatilities, green: at the money.

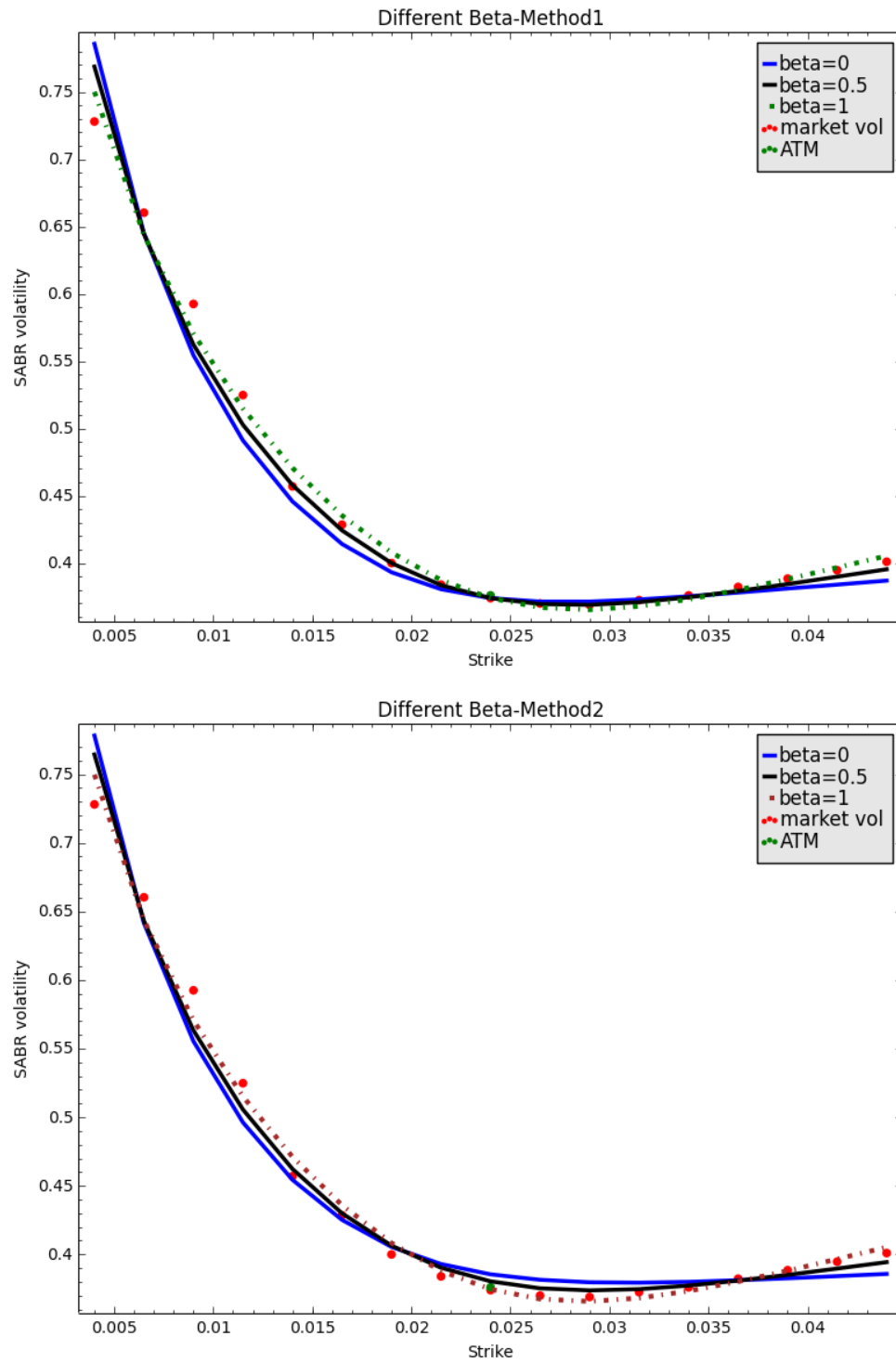


Figure 4.11: 1M5Y Calibration with different beta using Method 1 (top panel), 1M5Y Calibration with different beta using Method 2 (bottom panel). Blue: SABR volatility smile for $\beta = 0$, black: SABR volatility smile for $\beta = 0.5$, brown: SABR volatility smile for $\beta = 1$, red: market volatilities, green: at the money.

4.5.2 Method 1 vs. Method 2 using refinements

We compare the quality of fit between the two methods with their refinement suggested by (Oblój, 2008) for beta: 0, 0.5 and 1 .

| Calibration | beta | rho | volvol | alpha | error |
|-------------|------|---------|---------|-----------|----------|
| Method 1 | 0 | 0.64 | 0.31459 | 0.0071249 | 0.011801 |
| Method 2 | 0 | 0.68486 | 0.28987 | 0.0072634 | 0.011431 |
| Method 1F | 0 | 0.63942 | 0.31123 | 0.0071269 | 0.012125 |
| Method 2F | 0 | 0.6815 | 0.2879 | 0.0072645 | 0.011281 |

Table 4.4: Different methods calibrated when beta is 0

| Calibration | beta | rho | volvol | alpha | error |
|-------------|------|---------|---------|----------|-----------|
| Method 1 | 0.5 | 0.22619 | 0.2964 | 0.040393 | 0.0062754 |
| Method 2 | 0.5 | 0.22995 | 0.28271 | 0.04082 | 0.0064435 |
| Method 1F | 0.5 | 0.20306 | 0.29093 | 0.040465 | 0.0056859 |
| Method 2F | 0.5 | 0.20601 | 0.28021 | 0.040805 | 0.0057823 |

Table 4.5: Different methods calibrated when beta is 0.5

| Calibration | beta | rho | volvol | alpha | error |
|-------------|------|----------|---------|---------|-----------|
| Method 1 | 1 | -0.19683 | 0.33847 | 0.23802 | 0.0050049 |
| Method 2 | 1 | -0.20166 | 0.33447 | 0.23894 | 0.0051533 |
| Method 1F | 1 | -0.19683 | 0.33847 | 0.23802 | 0.0050049 |
| Method 2F | 1 | -0.20166 | 0.33447 | 0.23894 | 0.0051533 |

Table 4.6: Different methods calibrated when beta is 1

The results of the estimated parameters and errors are given in Tables 4.4, 4.5 and 4.6. Method 1 with refinement has the lowest error. Also, refinement in both methods fit well the market data. The calibrations are illustrated on Figures 4.12 and 4.13.

In Figure 4.12, a 5Y12Y swaption is calibrated using method 1 with and without fine-tuning (top panel) and using method 2 with and without fine-tuning (bottom panel). The fit of the two curves are almost identical for the entire domain of the curve and only seems to diverge slightly for large strikes. Similar results were observed with 1M5Y swaption (Figure 4.13).

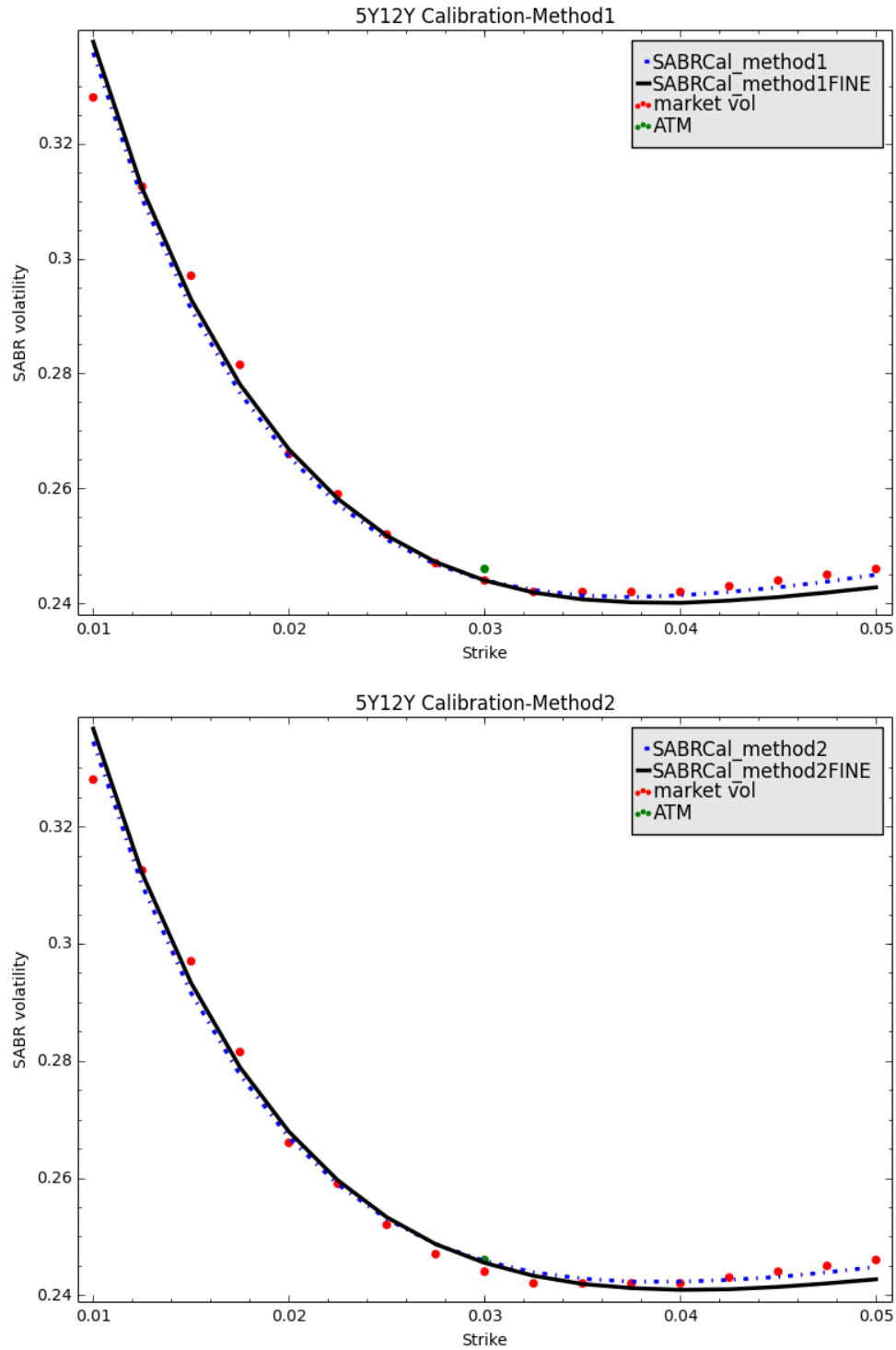


Figure 4.12: Top panel: 5Y12Y swaption calibration using Method 1, blue: SABR calibration using (Hagan *et al.*, 2002) formulae, black: SABR calibration using (Oblój, 2008) (refinement). Bottom panel: 5Y12Y swaption calibration using Method 2, blue: SABR calibration using (Hagan *et al.*, 2002) formulae, black: SABR calibration using (Oblój, 2008) (refinement). Red: market volatilities, green: at the money.

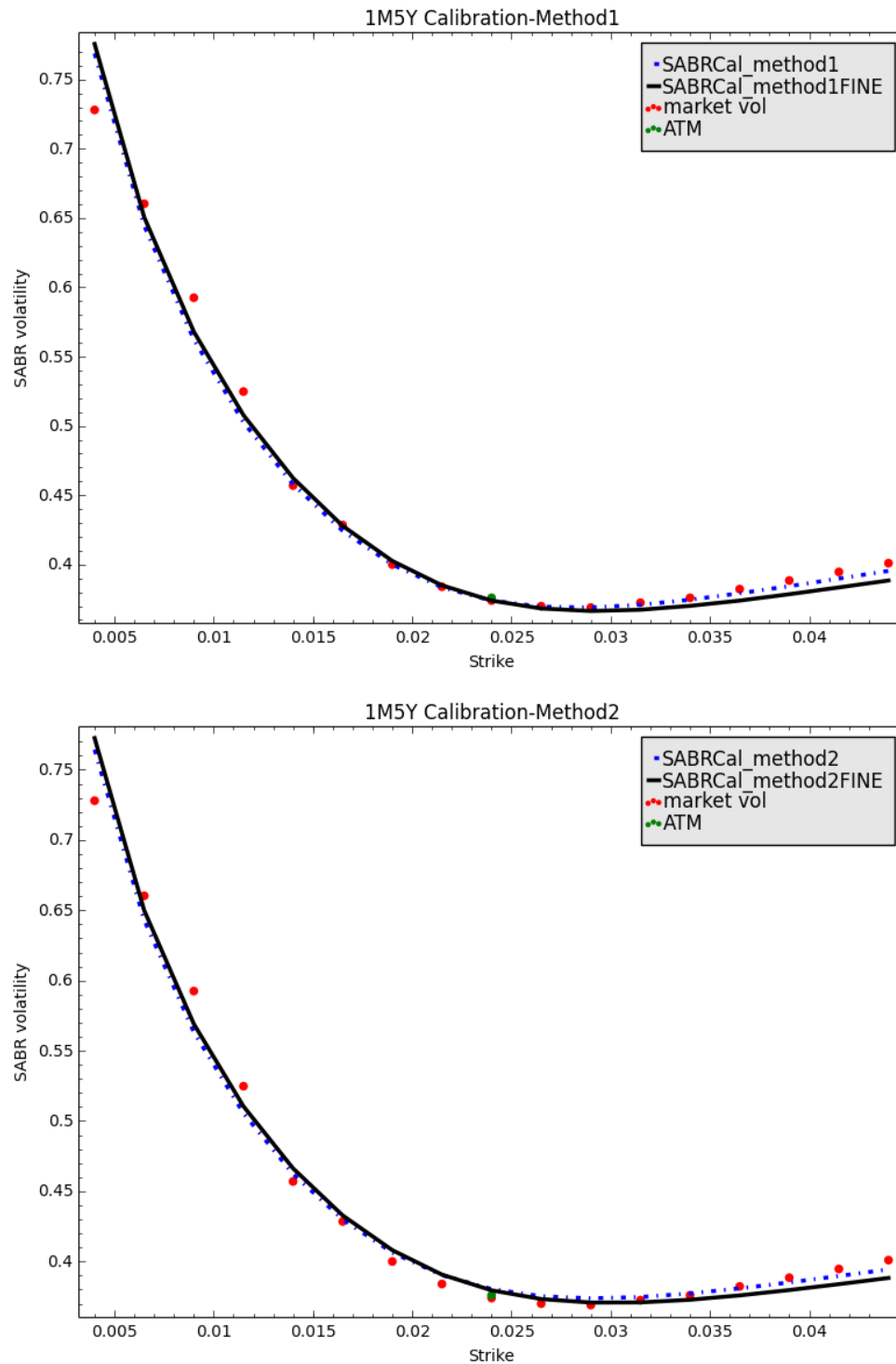


Figure 4.13: Top panel: 1M5Y swaption calibration using Method 1, blue: SABR calibration using (Hagan *et al.*, 2002) formulae, black: SABR calibration using (Oblój, 2008) (refinement). Bottom panel: 1M5Y swaption calibration using Method 2, blue: SABR calibration using (Hagan *et al.*, 2002) formulae, black: SABR calibration using (Oblój, 2008) (refinement). Red: market volatilities, green: at the money.

4.5.3 Calibration for long and short tenors

For both methods and for a chosen beta, we have seen that the SABR model has a good fitness to a 1M5Y swaptions and 5Y12Y swaptions. Other combinations of swaptions are possible, for this reason we look at the following swaptions: 1M4Y, 20Y4Y, 1M20Y, 20Y20Y.

In Figures 4.14 and 4.15, the SABR model is calibrated for a 1M4Y and a 20Y4Y swaption (in this order) for $\beta = 0.5$ and $\beta = 1$. In Figure 4.16 the SABR model is calibrated for a 1M20Y and a 20Y20Y swaption for $\beta = 0.5$. The calibrated parameters: ρ, ν, α as well as **sse** can be found in Table 4.7. The 1M4Y and 20Y4Y swaptions for $\beta = 0.5$ (top panels of Figures 4.14 and 4.15) do not have a good fitness. The fitness is even horrible with $\beta = 0$. In this case we did not report the figure, nevertheless, we can see how big is the error in Table 4.7. However, we can see a “quite” good fitness when $\beta = 1$ (bottom panels of Figures 4.14 and 4.15). In contrast, the 1M20Y and 20Y20Y swaptions for $\beta = 0.5$ do have a very good fitness (Figure 4.16). We underline that, the choice of beta in some scenarios is crucial as we have seen with the 1M4Y and 20Y4Y swaptions. Another way to get a good fitness maybe to try with refinements. This can be seen as a proof of the complexity and dynamic the market.

The SABR model in (Hagan *et al.*, 2002) was developed before the credit crunch (interest rates were still somewhat high). But after the crisis, the dynamics of interest rates have changed, as we have seen in Chapter 3. The market has witnessed scenarios such as: negative interest rate and negative strikes! The SABR model breaks down and we can no longer calibrate the curve under methods presented so far, consequently we need methods that account for negative strikes.

| Swaption | alpha | beta | rho | volvol | error |
|----------|-----------|------|-----------|----------|-----------|
| 1M4Y | 0.0080242 | 0 | 0.53584 | 0.51626 | 0.05425 |
| 1M5Y | 0.008953 | 0 | 0.447103 | 0.651421 | 0.024757 |
| 1M20Y | 0.0091167 | 0 | 0.22047 | 0.73638 | 0.012177 |
| 20Y4Y | 0.0049645 | 0 | 1 | 0.26013 | 0.13542 |
| 20Y20Y | 0.0048115 | 0 | 0.39135 | 0.17722 | 0.0094705 |
| 1M4Y | 0.053886 | 0.5 | 0.089563 | 0.54186 | 0.042698 |
| 1M5Y | 0.057738 | 0.5 | 0.08426 | 0.673686 | 0.014563 |
| 1M20Y | 0.05122 | 0.5 | -0.034454 | 0.76423 | 0.0054791 |
| 20Y4Y | 0.031644 | 0.5 | 1 | 0.061883 | 0.015523 |
| 20Y20Y | 0.026842 | 0.5 | -0.1098 | 0.20861 | 0.0050551 |
| 1M4Y | 0.36249 | 1 | -0.27541 | 0.67734 | 0.025463 |
| 1M5Y | 0.373115 | 1 | -0.24621 | 0.783108 | 0.014482 |
| 1M20Y | 0.28813 | 1 | -0.26626 | 0.83733 | 0.0082975 |
| 20Y4Y | 0.23422 | 1 | -0.49579 | 0.12601 | 0.011399 |
| 20Y20Y | 0.16128 | 1 | -0.42515 | 0.2771 | 0.0058998 |

Table 4.7: Some swaptions calibrated with Method 1 of (Hagan *et al.*, 2002).

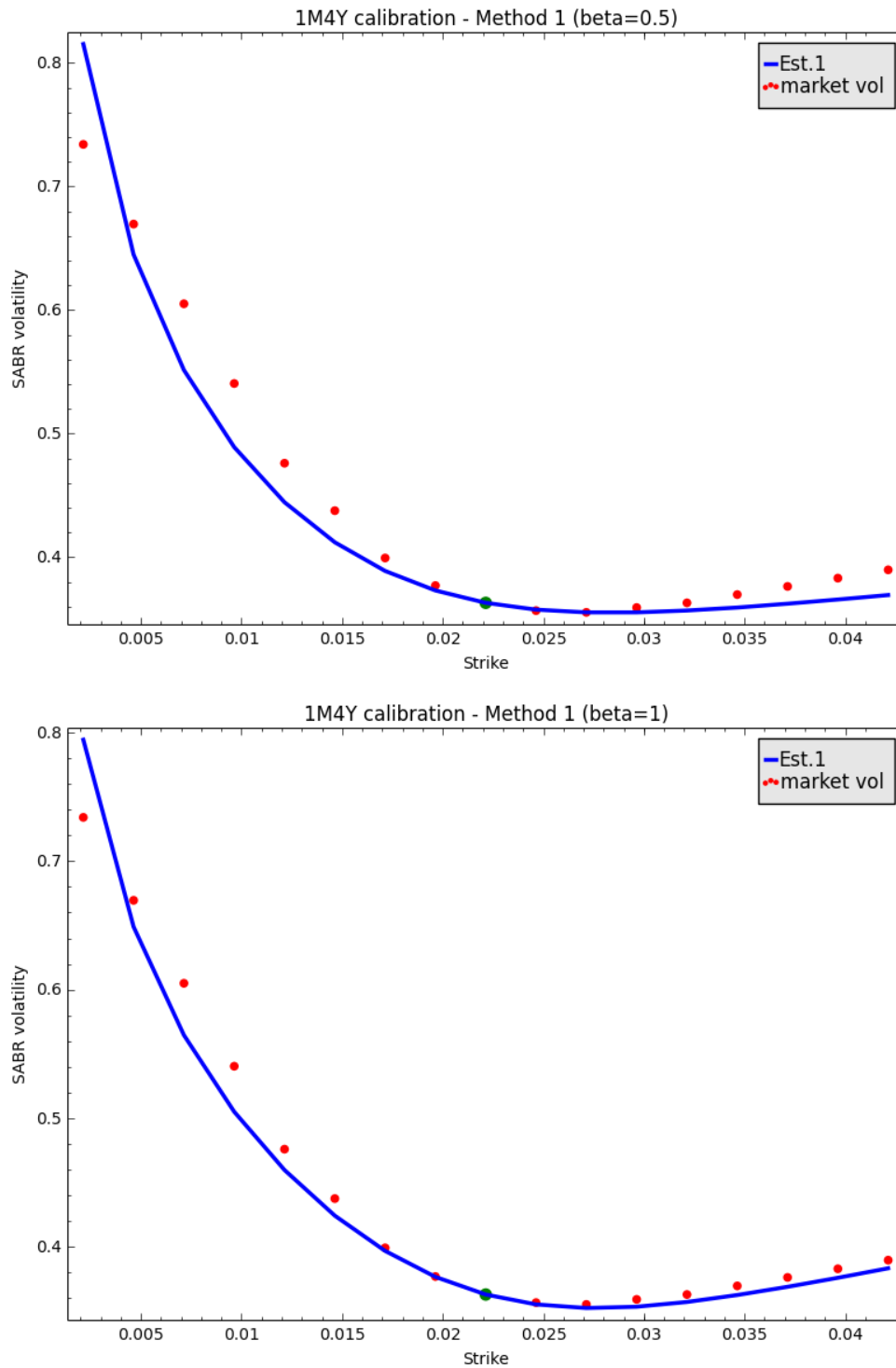


Figure 4.14: Top panel: 1M4Y swaption calibration using Method 1 for $\beta = 0.5$. Bottom panel: 1M4Y swaption calibration using Method 1 for $\beta = 1$. Blue: SABR calibration using (Hagan *et al.*, 2002) formulae, red: market volatilities, green: at the money.

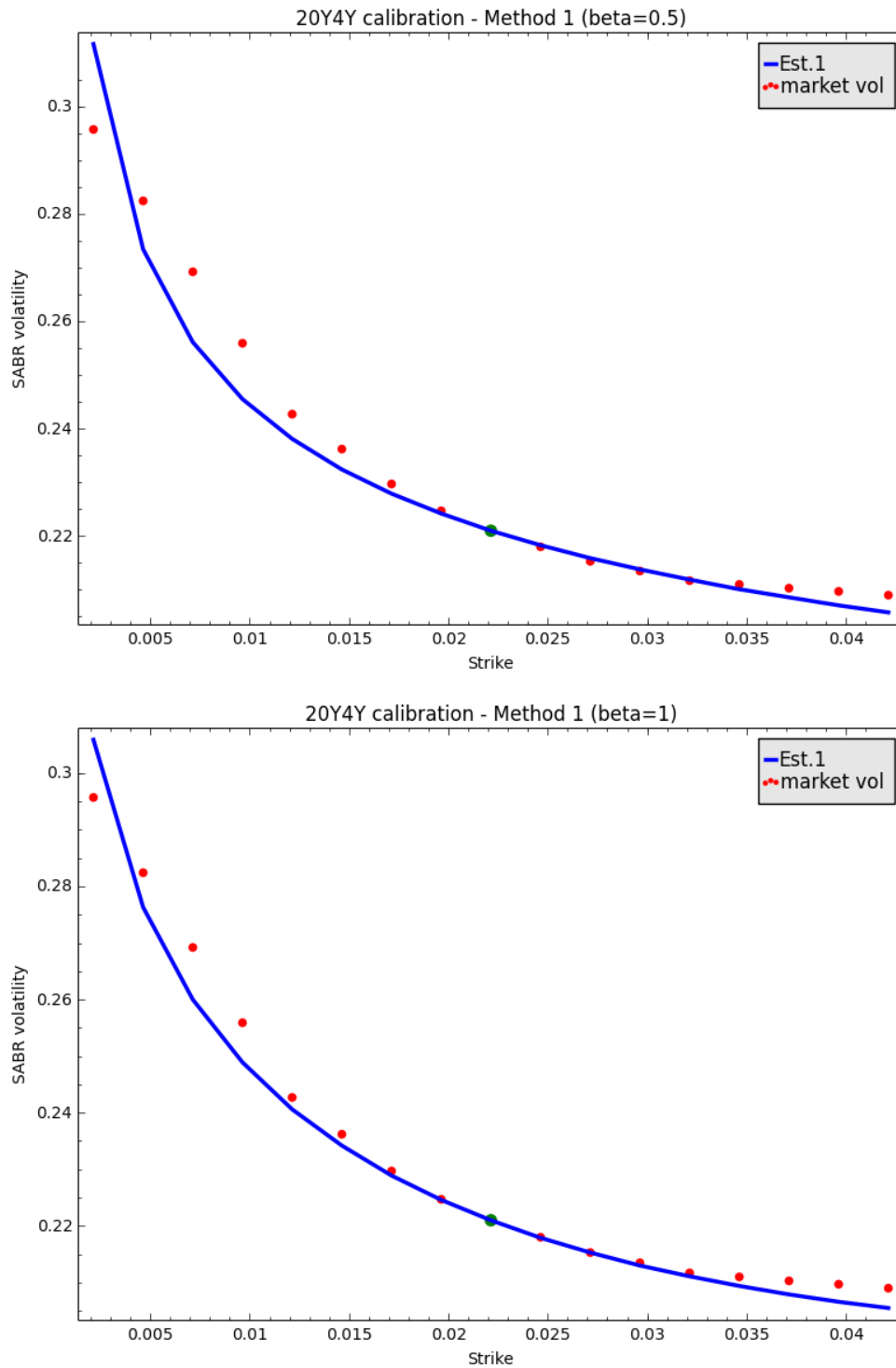


Figure 4.15: Top panel: 20Y4Y swaption calibration using Method 1 for $\beta = 0.5$. Bottom panel: 20Y4Y swaption calibration using Method 1 for $\beta = 1$. Blue: SABR calibration using (Hagan *et al.*, 2002) formulae, red: market volatilities, green: at the money.

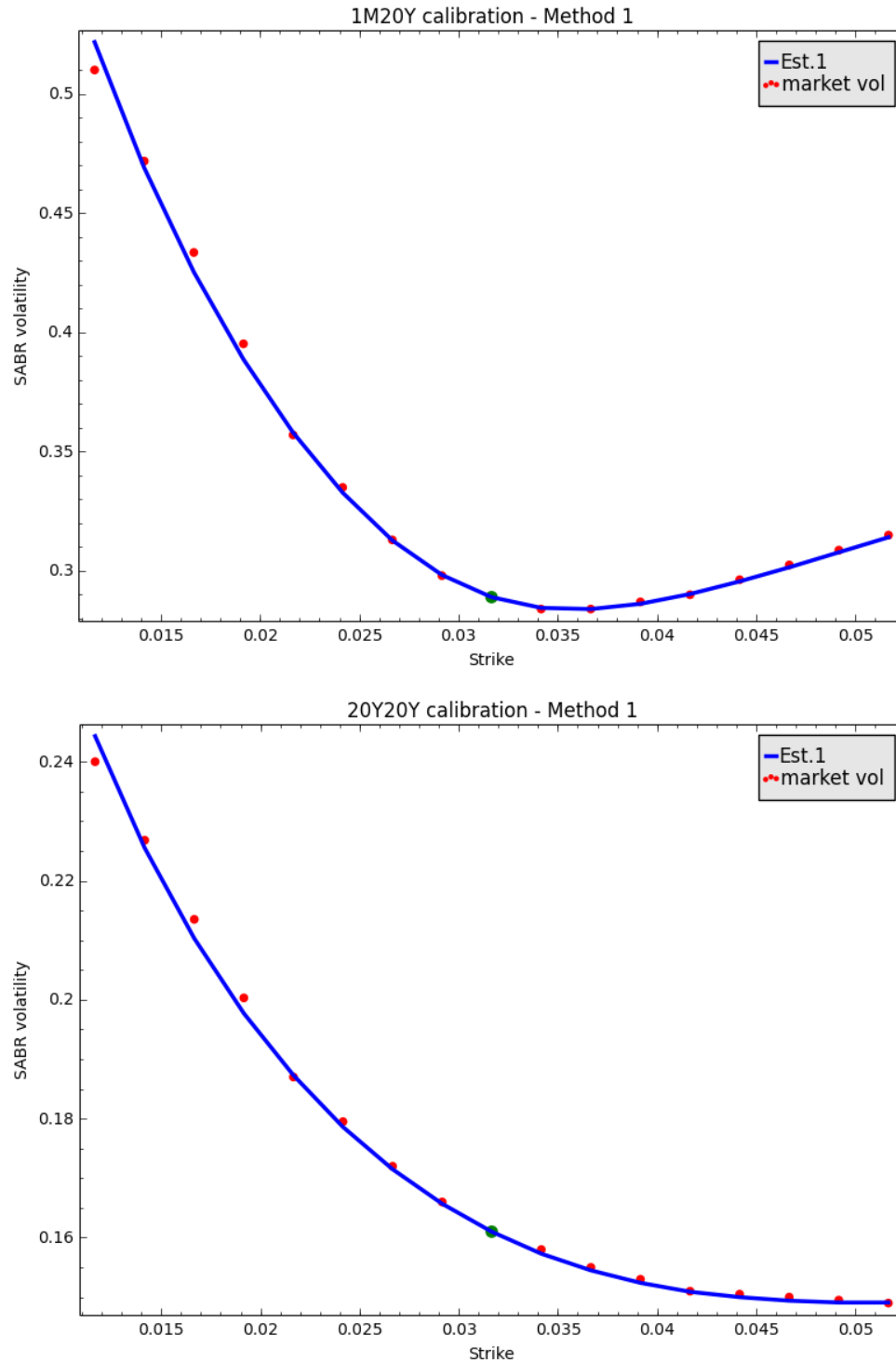


Figure 4.16: Top panel: 1M20Y swaption calibration using Method 1 for $\beta = 0.5$. Bottom panel: 20Y20Y swaption calibration using Method 1 for $\beta = 0.5$. Blue: SABR calibration using (Hagan *et al.*, 2002) formulae, red: market volatilities, green: at the money.

4.6 Summary and conclusion

In this chapter, we have reviewed and calibrated the SABR model for different swaptions. We have used two different methods (with and without refinement), we have found that the SABR model accurately capture the volatility smiles on the markets. There is no considerable difference between both methods. We have found that refinements (produce smaller errors) may be good candidate for a better fitness. We have also seen that, in some scenarios, the claim that “the choice of the parameter β has no effect on the smile” is not true in general.

This work can be extended to studying how to calculate swaptions, caps or floors prices, and how they are affected by changing the parameters under the SABR model ³. Moreover, an extension of the SABR model that is consistent for negative strikes is needed.

³this is the object of the next chapter (Chapter 5)

Chapter 5

Interest rate curves and SABR calibration applications

In this chapter, we consider problems that use interest rate curves and SABR Calibration. Particularly, we consider a method for transforming the market observed smile to non-standard tenors by applying the SABR model for a robust bootstrap of the whole smile. We use the results described in Chapters 2, 3 and 4.

5.1 Introduction

Assuming that information on volatility smiles is provided by the market directly through quotes of swaption smiles. Due to the fact that not every swaption in the standard at-the-money matrix has also quotes for away-from-the-money strikes and when quotes are often too few to allow for a robust bootstrap of the whole smile; the following question, therefore, arise naturally in the current interest-rate volatility market:

- what is the value of implied volatility for strikes outside (and in particular larger than) the quoted ones?

To answer such question rigorously, one should in principle resort to a dynamical interest-rate model. Assuming, for instance, specific forward-rate dynamics, as in the Libor market models of (Piterbarg, 2003), (Errais *et al.*, 2004) or (Henry-Labordere, 2006), one can employ suitable approximations for swaption prices calibrate the market swaption smiles. With the calibrated parameters, one can then price swaptions with any strike. However, even neglecting the issue of accuracy of the approximations, this approach may be too time-consuming and does not necessarily grant the precision required for a good estimate of the missing volatility values. In fact, model parameters are typically too many for the calibration to be fast and, at the same time, too few to recover (almost) exactly the quoted market data. One can thus prefer

to resort to a less elegant solution, which however answers the above question in a rapid and consistent way.

In this chapter, we will then propose an empirical, yet consistent, procedure that serves this purpose. To this end, we will use the functional form for implied volatility derived by Hagan *et al.* (2002) which has been described in Chapter 4. This work was partly done by Mercurio and Pallavicini (2006) in the single curve framework, in this chapter we extend the work in the multi-curve framework and this constitute one of our main contribution. This result shows how to transform the market observed smile to non-standard tenors by applying the SABR model for a robust bootstrap of the whole smile. This shows as well the robustness of the SABR model to calibrate the market volatility smile coherently with the new market evidences as did Bianchetti and Carlicchi (2011).

5.2 Swaption smile and SABR functional form

We consider, in the same chapter 2, the ATM strike defined in equation (2.3.5) as the forward swap rate defined in equation (2.5.6). Therefore, we can adapt the result got in chapter 4 as follows.

Under the T -forward measure, the SABR model is described by the following system of stochastic differential equations:

$$dS(t, k, n) = \hat{\alpha} [S(t, k, n)]^\beta dW_1, \quad (5.2.1)$$

$$d\hat{\alpha} = \nu \hat{\alpha} dW_2, \quad \hat{\alpha}(0) = \alpha \quad (5.2.2)$$

$$\mathbb{E}[dW_1 dW_2] = \rho dt \quad (5.2.3)$$

where, $S(0, k, n)$ is ATM strike, $\hat{\alpha}$ is the volatility, and W_1 and W_2 are two correlated Wiener processes. The implied volatility $\sigma_B(S(0, k, n), K)$ is approximately given by:

$$\sigma_B(S(0, k, n), K) \simeq U \left(\frac{z}{X(z)} \right) V \quad (5.2.4)$$

where

$$U = \alpha \left\{ [S(0, k, n)K]^{(1-\beta)/2} \left[1 + \frac{(1-\beta)^2}{24} \log^2 \left(\frac{S(0, k, n)}{K} \right) + \frac{(1-\beta)^4}{1920} \log^4 \left(\frac{S(0, k, n)}{K} \right) \right] \right\}^{-1} \quad (5.2.5)$$

$$V = \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{[S(0,k,n)K]^{1-\beta}} + \frac{\rho\beta\nu\alpha}{4[S(0,k,n)K]^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2 \right] t_{ex} + \dots \right\}. \quad (5.2.6)$$

Here

$$z = \frac{\nu}{\alpha} [S(0, k, n)K]^{(1-\beta)/2} \log \left(\frac{S(0,k,n)}{K} \right) \quad (5.2.7)$$

and $X(z)$ is defined by

$$X(z) = \log \left\{ \frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho} \right\}. \quad (5.2.8)$$

Where $S(0, k, n)$ is given by

$$S(0, k, n) = \frac{P(0, T_k) - P(0, T_n)}{\sum_{i=k+1}^n \delta(T_{i-1}, T_i) P(0, T_i)}. \quad (5.2.9)$$

Formula (5.2.4) provides us with an (efficient) approximation for the SABR implied volatility for each strike K . Its simplicity, combined with the intuitive meaning of model parameters, has made it a popular choice for swaption smile analysis, to the point of becoming a market standard for modelling implied volatility (this happened despite the fact that such approximation can lead to arbitrage opportunities for long maturities and large strike values).

We emphasize that, the procedure of calibration remains the same we did in Chapter 4. The only difference being in substituting the forward rate f in Chapter 4 by $S(0, k, n)$ in equation (5.2.9). However, $S(0, k, n)$ in equation (5.2.9) requires the construction of the discount curve, and this was the object of Chapter 2. We have to underline that changing f by $S(0, k, n)$ in the calibration process will slightly affect the values of the SABR parameters as we will see in the example below.

In the multi-curve framework, as we have presented in Chapter 3, the procedure is the same as above (formulae (5.2.1) to (5.2.8) remain valid). The only difference being in finding $S(0, k, n)$. In the multi-curve framework, $S(0, k, n)$ is given by

$$S(0, k, n) = \frac{\sum_{i=k}^n \delta(T_{i-1}, T_i) P_c(0, T_i) F(0; T_{i-1}, T_i)}{\sum_{i=1}^n \delta(T_{i-1}, T_i) P_c(0, T_i)}. \quad (5.2.10)$$

$S(0, k, n)$ in equation (5.2.10) requires the construction of the forward curve and discount curve, and this was treated in Chapter 3.

Example 5.1. *This example shows to calibrate the multi-curve data for volatility smiles. This is in the same way a methodology for transferring the shape of the smile to non-standard tenors. To this end we have chosen the SABR*

parameters $\beta = 0.75$, $\rho = 0.47$, $\nu = 0.76$ and $T = 3.5$. The forward swap rates are $S(t, k, n) = 0.000792831267250977$ (in the multi-curve) and $S(t, k, n) = 0.0252819292623$ (in the single-curve). We take the volatility $\sigma = 0.6743$. Once we have the ATM volatility derived by the transformation method we calculate the parameter α to determine a SABR model with the new transformed volatility. The corresponding SABR parameters are calculated using equation (3.3.7) and are given by $\alpha = 0.0912330160542767$, $\alpha = 0.2167334588286166$. As expected in Figure 4.6, the shape of the smile is preserved in terms of monyness. After transforming the whole smile for non-standard tenors we are able to calculate swaptions, caps or floors prices, and how they are affected by changing the parameters under the SABR model¹ as we have announced in Chapter 4.

5.3 Summary and conclusion

In this chapter, we have proposed an extension of a procedure for stripping consistently implied volatilities in the multi-curve framework. We have highlighted the procedure of the SABR calibration after the crisis. We claim that, after the crisis, the calibration procedure of the SABR model remain the same as before the crisis. However the values of the parameters may vary (Example 5.1 show the case in the value of the parameter α).

¹formulae for this purpose are presented in the previous chapters

Chapter 6

Conclusion

In Chapter 2, we have introduced the methodologies for construction interest rate curves. We have reviewed the fundamental pricing formulas for plain vanilla interest rate derivatives in the classical framework with no collateral. Also, we have implemented the methodologies (Bootstrapping and Best Fit) for constructing discounting and forward yield curves.

In Chapter 3, we have given an overview of the changes that have taken place in the financial markets since the credit crunch of 2007. No longer can we use single-curve models when pricing and hedging plain vanilla single-currency interest rate derivatives. Instead, we use multiple distinct curves to ensure market coherent estimation of discount factors and of forward rates with different underlying rate tenors. We have reviewed the fundamental pricing formulas for plain vanilla interest rate derivatives, extending the single-curve framework to the modern market situation with collateral.

In Chapter 4, we have reviewed and calibrated the SABR model for different swaptions. We have used two different methods (with and without refinement), we have found that the SABR model accurately capture the volatility smiles on the markets. There is no considerable difference between both methods. We have found that refinements may be a good candidate for a better fitness. We have also seen that, in some scenarios, the choice of the parameter beta has effect on the smile. In addition, from the same source of data, we have seen that the value of the parameter beta that procedure a “good” fitness is not the same for all the swaptions (this shows the complexity of the market after the credit crunch).

In Chapter 5, we have proposed an extension of a procedure for stripping consistently implied volatilities in the multi-curve framework. Further work can extend the procedure to strip CMS swap spreads. We have highlighted the procedure of the SABR calibration after the crisis.

On the whole, we can see that the SABR functional form can calibrate consistently implied volatilities in the multi-curve framework. Furthermore, it can well interpolate and extrapolate swap-rate volatility data and it is an effective tool for pricing simple derivatives such as caps and floors, which are

very sensitive to the swaption smile, without resorting to a fully-consistent interest-rate model.

Finally, the work presented in this thesis can be extended in many directions. We need a reconstruction of the interest rate modelling framework in a post-crisis financial world, with Libor rates incorporating credit and liquidity risks. We need to include the cost of funding in constructing the OIS discounting yield curve and we need to include basis spreads in constructing of multiple forwarding yield curves.

Another extension will be to describe separately the discount and forward yield curves using general short term interest rate models such as: [Ho and Lee \(1986\)](#), [Hull and White \(1993\)](#), [Kalotay, Williams and Fabozzi \(1993\)](#), [Black and Karasinski \(1991\)](#), [Black, Derman and Toy \(1990\)](#), etc.

The short term interest rate models for the discount and forward interest rates need not to be the same in general. We can then approximate continuous time models on discrete lattice, calculate interest rate trees (binomial or trinomial) for both discount and forward rates and calculate the cash flows tree using both discount and forward rates. This approach (proposed by [Oleksandr et al. \(2014\)](#)) can be used for pricing any interest rate derivative. In addition, this approach can be extended by considering affine jump diffusion interest rate models or lévy term structure models.

We need to compute the delta sensitivity of a portfolio consisting of instruments with a variety of different tenors, the convexity and other Greeks. It is important to develop the model which can handle multiple dynamic curves and its practical calibration scheme.

Appendices

Appendix A

Numéraire change

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a filtered probability space describing the market.

Definition A.1 (Numéraire). *A numéraire is a particular tradable asset or a tradable economic entity that can be used to express the relative prices of all other tradables in terms of its price. The relative price (relative to N) for any security S , for a given numéraire asset N , is defined by:*

$$\frac{S}{N}.$$

A numéraire must have strictly positive value, and must not have intermediate cash payments. There are many assets (or portfolios) that could be chosen as a numéraire. A numéraire is chosen for convenience only, and for no other reason. The choice of numéraire does not affect eventual prices in any way.

Definition A.2 (Equivalent Martingale Measure (EMM)). *A measure \mathbb{Q}^N is an Equivalent Martingale Measure (EMM) relative to the numéraire N if*

1. \mathbb{Q}^N is equivalent to the probability measure \mathbb{P} i.e. they have the same null sets

$$\mathbb{P}(X) = 0 \quad \text{if and only if} \quad \mathbb{Q}^N(X) = 0 \quad \text{for any } X \in \mathcal{F}$$

2. \mathbb{Q}^N is a martingale measure relative to N i.e. all relative prices are martingales under \mathbb{Q}^N

$$\frac{S(t)}{N_t} = \mathbb{E}^{\mathbb{Q}^N} \left[\frac{S(T)}{N_T} \middle| \mathcal{F}_t \right]. \quad (\text{A.0.1})$$

We assume that the market is a well-functioning market, in other words, there is no arbitrage opportunities in the market¹.

¹Arbitrage opportunities represent the limitless creation of wealth through risk-free profit

Based on this assumption, we use the result “Market is free of arbitrage if and only if there exists an EMM (say \mathbb{Q}^Y)” established by (Harrison and Kreps, 1979), (Harrison and Pliska, 1981) and (Harrison and Pliska, 1983) .

The natural question one may ask now is: what is this EMM \mathbb{Q}^Y ?

Proposition A.3. *Since the choice of numéraire is irrelevant (we refer to (Ekstrand, 2011) for more details), if there exists an EMM \mathbb{Q}^N relative to numéraire N then for any other numéraire M there exists an EMM \mathbb{Q}^M relative to M .*

From this proposition, we can deduce that \mathbb{Q}^Y is equivalent to the EMM \mathbb{Q}^B (corresponding to the bank account from Definition 2.2) often called the risk-neutral measure. Hence if $P(t, T)$ (Definition 2.3) is taken as the numéraire (i.e. $N_t = P(t, T)$), then using formula (A.0.1), we have $\mathbb{Q}^N = \mathbb{Q}^{T^2}$ and

$$S(t) = P(t, T) \mathbb{E}^{\mathbb{Q}^T} [S(T) | \mathcal{F}_t]. \quad (\text{A.0.2})$$

Similarly, if $P_c(t, T)$ is the numéraire (i.e. $N_t = P_c(t, T)$), then we have

$$S(t) = P_c(t, T) \mathbb{E}^{\mathbb{Q}_c^T} [S(T) | \mathcal{F}_t]. \quad (\text{A.0.3})$$

Formulae (A.0.2) and (A.0.3) are tools for pricing, we use Formula (A.0.2) in the market with no-collateralisation (before the crisis) and (A.0.3) in the market with collateralisation (after the crisis, where the subscript c stands for under CSA).

²**discounting** T_j -forward measure \mathbb{Q}^{T_j} , associated to the numéraire $P(t, T_j)$

Appendix B

Day Count Conventions

The different types of day count are: Actual/Actual, Actual/365, Actual/360 and 30/360.

- Actual/Actual: (Actual/365, Actual/360) means that the accrued interest between the two given dates is calculated using the exact number of calendar days between the two dates divided by the exact number of calendar days of the ongoing year (365, 360). For example, using the Actual/Actual or Actual/365 day-count basis, there are 65 days from 04/03/2013 to 06/07/2013. Using the Actual/365 day-count basis, the period from 03/07/2011 to 04/09/2013 converted in years is

$$\frac{299}{365} + 1 + \frac{99}{365} = \frac{763}{365} = 2.090.$$

Using the Actual/360 day-count basis, the period from 03/07/2011 to 04/09/2013 converted in years is

$$\frac{763}{360} = 2.1194.$$

- 30/360: means that the number of the calendar days between the two dates is computed assuming that each month count as 30. For instance, there are 75 days ($2 \times 30 + 15$) from 02/04/2013 to 04/19/2013 and 277 days ($9 \times 30 + 7$) from 03/11/2013 to 12/18/2013. Using the 30/360 day-count basis, the period from 03/07/2011 to 04/09/2013 converted in years is

$$\frac{25 \times 30 + 2}{360} = 2.0888.$$

Appendix C

Interpolation and LMA

This appendix presents some interpolation methods and the Levenberg-Marquardt algorithm.

C.1 Linear interpolation

Let $P := \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$ be a partition of the interval $[a, b]$.

Definition C.1. A Linear Interpolation (function) L_P on L is a real function $L_P : [a, b] \rightarrow \mathbb{R}$ with the property: L_P coincides on every subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$, with a polynomial of degree one.

Let $Y := \{y_0, y_1, \dots, y_n\}$, $L_P(Y; \cdot)$ an linear Interpolation function L_P with $L_P(Y; x_i) = y_i$ for $i = 0, 1, \dots, n$ and $h_{i+1} = x_{i+1} - x_i$, for $i = 0, 1, \dots, n-1$. We prove that

$$L_P(Y; x) = y_i \frac{x_{i+1} - x}{h_{i+1}} + y_{i+1} \frac{x - x_i}{h_{i+1}} \quad \text{for } x \in [x_i, x_{i+1}].$$

Linear interpolation is often used to approximate a value of some function G using two known values of that function at other points (for instance $x = u$ and $x = v$). The error of this approximation is defined as

$$R_T = G(x) - L_P(Y; x)$$

By virtue of It can be proven using Rolle's theorem¹ we can prove that:

$$L_P(Y; x) = G(u) + \frac{G(v) - G(u)}{v - u}(x - u).$$

¹ **Rolle's theorem** If a real-valued function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b) , and $f(a) = f(b)$, then there exists a c in the open interval (a, b) such that $f'(c) = 0$.

with the error term bounded by:

$$|R_T| \leq \frac{(u-v)^2}{8} \max_{u \leq x \leq v} |G''(x)|.$$

Linear interpolation can be applied on rates, logarithm of rates, discount factors, logarithm of discount factor and on forward rates.

C.2 Cubic splines

Let $P := \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$ be a partition of the interval $[a, b]$.

Definition C.2. A cubic spline (function) S_P on P is a real function $S_P : [a, b] \rightarrow \mathbb{R}$ with the properties:

- (1) $S_P \in C^2[a, b]$, that is, S_P is twice continuously differentiable on $[a, b]$.
- (2) S_P coincides on every subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$, with a polynomial of degree three.

Let $Y := \{y_0, y_1, \dots, y_n\}$, $S_P(Y; \cdot)$ an interpolating spline function S_P with $S_P(Y; x_i) = y_i$ for $i = 0, 1, \dots, n$ and $h_{i+1} = x_{i+1} - x_i$, for $i = 0, 1, \dots, n-1$.

Such S_P is not unique, the following three conditions by itself ensures uniqueness of S_P (it has been proved in (Bulirsch and Stoer, 2002))

- (i) $S_P''(Y; a) = S_P''(Y; b) = 0$,
- (ii) $S_P^m(Y; a) = S_P^m(Y; b) = 0$, for $m = 0, 1, 2$: $S_P(Y; \cdot)$ is periodic,
- (iii) $S_P'(Y; a) = \gamma_0$, $S_P'(Y; b) = \gamma_n$, for given γ_0, γ_n .

Let the values of the second derivatives at $x_i \in P$, be defined by

$$M_i := S_P''(Y; x_i), \quad i = 0, 1, \dots, n. \quad (\text{C.2.1})$$

Property (2) in definition C.2, implies that $S_P''(Y; \cdot)$ is a linear function in each $[x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$. Therefore, $S_P''(Y; \cdot)$ can be written as

$$S_P''(Y; x) = M_i \frac{x_{i+1} - x}{h_{i+1}} + M_{i+1} \frac{x - x_i}{h_{i+1}} \quad \text{for } x \in [x_i, x_{i+1}].$$

By successive integration, for $x \in [x_i, x_{i+1}]$ and $i = 0, 1, \dots, n-1$, we have

$$\begin{aligned} S_P'(Y; x) &= -M_i \frac{(x_{i+1} - x)^2}{2h_{i+1}} + M_{i+1} \frac{(x - x_i)^2}{2h_{i+1}} + A_i, \\ S_P(Y; x) &= M_i \frac{(x_{i+1} - x)^3}{6h_{i+1}} + M_{i+1} \frac{(x - x_i)^3}{6h_{i+1}} + A_i(x - x_i) + B_i. \end{aligned} \quad (\text{C.2.2})$$

Using the fact $S_P(Y; x_i) = y_i$ and $S_P(Y; x_{i+1}) = y_{i+1}$, we obtain

$$\begin{aligned} B_i &= y_i - M_i \frac{h_{i+1}^2}{6}, \\ A_i &= \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{6} (M_{i+1} - M_i). \end{aligned} \quad (\text{C.2.3})$$

Substituting (C.2.3) in (C.2.2) yields

$$S'_P(Y; x) = -M_i \frac{(x_{i+1} - x)^2}{2h_{i+1}} + M_{i+1} \frac{(x - x_i)^2}{2h_{i+1}} + \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{6} (M_{i+1} - M_i).$$

For $j = 1, 2, \dots, n-1$, we have therefore

$$\begin{aligned} S'_P(Y; x^-) &= \frac{y_i - y_{i-1}}{h_i} + \frac{h_i}{3} M_i + \frac{h_i}{6} M_{i-1}, \\ S'_P(Y; x^+) &= \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{3} M_i - \frac{h_{i+1}}{6} M_{i+1}. \end{aligned}$$

The continuity of $S'_P(Y; \cdot)$ at $x = x_i$ implies that $S'_P(Y; x^-) = S'_P(Y; x^+)$, hence for $j = 1, 2, \dots, n-1$ we have

$$\frac{h_i}{6} M_{i-1} + \frac{h_i + h_{i+1}}{3} M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}. \quad (\text{C.2.4})$$

Using the conditions (i), (ii) and (iii) above, we have

- Case(i):

$$S''_P(Y; a) = M_0 = M_n = S''_P(Y; b).$$

- Case(ii):

$$\begin{aligned} S''_P(Y; a) &= S''_P(Y; b) \implies M_0 = M_n; \\ S'_P(Y; a) &= S'_P(Y; b) \implies \frac{h_n}{6} M_{n-1} + \frac{h_n + h_1}{3} M_n + \frac{h_1}{6} M_1 \\ &= \frac{y_1 - y_n}{h_1} - \frac{y_n - y_{n-1}}{h_n}. \end{aligned}$$

The latter condition is identical to (C.2.4) for $i = n$ if we put

$$h_{n+1} := h_1, \quad M_{n+1} := M_1, \quad y_{n+1} := y_1.$$

- Case(iii):

$$\begin{aligned} S'_P(Y; a) &= y'_0 \implies \frac{h_1}{3} M_0 + \frac{h_1}{6} M_1 = \frac{y_1 - y_0}{h_1} - y'_0. \\ S'_P(Y; b) &= y'_n \implies \frac{h_n}{6} M_{n-1} + \frac{h_n}{3} M_n = y'_n - \frac{y_n - y_{n-1}}{h_n} \end{aligned}$$

The last two equations and those in (C.2.4) can be written as

$$\mu_i M_{i-1} + 2M_i + \lambda_j M_{i+1} = d_i, \quad j = 1, 2, \dots, n-1,$$

where

$$\begin{aligned} \lambda_i &:= \frac{h_{i+1}}{h_i + h_{i+1}}, \quad \mu_i := 1 - \lambda_i = \frac{h_i}{h_i + h_{i+1}} \\ d_i &:= \frac{6}{h_i + h_{i+1}} \left\{ \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \right\} \end{aligned}$$

In case (i), we define in addition

$$\lambda_0 := 0, \quad d_0 := 0, \quad \mu_n := 0, \quad d_n := 0,$$

in case (iii)

$$\begin{aligned} \lambda_0 &:= 1, \quad d_0 := \frac{6}{h_1} \left(\frac{y_1 - y_0}{h_1} - y'_0 \right) \\ \mu_n &:= 1, \quad d_n := \frac{6}{h_n} \left(y'_n - \frac{y_n - y_{n-1}}{h_n} \right). \end{aligned}$$

finally in case (ii), we define

$$\begin{aligned} \lambda_n &:= \frac{h_1}{h_n + h_1}, \quad \mu_n := 1 - \lambda_n = \frac{h_n}{h_n + h_1}, \\ d_i &:= \frac{6}{h_n + h_1} \left\{ \frac{y_1 - y_n}{h_1} - \frac{y_n - y_{n-1}}{h_n} \right\}. \end{aligned}$$

C.3 Levenberg-Marquardt algorithm

The Levenberg-Marquardt algorithm is an iterative procedure used to solve non-linear least squares problems (especially in least squares curve fitting). The Levenberg-Marquardt curve-fitting method is a combination of two minimization methods: the Gauss-Newton algorithm (GNA) and the gradient descent method.

The problem

Given a set of n empirical datum pairs of independent and dependent variables, (x_i, y_i) , optimize the parameters β of the model curve $f(x, \beta)$ (where f is non-linear in β) so that the sum of the squares of the deviations

$$S(\beta) = \sum_{i=1}^m [y_i - f(x_i, \beta)]^2 \tag{C.3.1}$$

becomes minimal.

The solution

To start a minimization, the user has to provide an initial guess for the parameter vector, β . In cases with only one minimum, an uninformed standard guess like $\beta^T = (1, 1, \dots, 1)$ will work fine; in cases with multiple minima, the algorithm converges only if the initial guess is already somewhat close to the final solution.

In each iteration step, the parameter vector, β , is replaced by a new estimate, $\beta + \delta$. To determine δ , the functions $f(x_i, \beta + \delta)$ are approximated by their linearizations

$$f(x_i, \beta + \delta) \approx f(x_i, \beta) + J_i \delta \quad (\text{C.3.2})$$

where

$$J_i = \frac{\partial f(x_i, \beta)}{\partial \beta}.$$

Substituting (C.3.2) in (C.3.1) gives

$$S(\beta + \delta) \approx \sum_{i=1}^n (y_i - f(x_i, \beta) - J_i \delta)^2.$$

Or in vector notation,

$$S(\beta + \delta) \approx \|\mathbf{y} - \mathbf{f}(\beta) - \mathbf{J}\delta\|^2. \quad (\text{C.3.3})$$

The minimum of the (C.3.3) is found by solving the equation: the derivative with respect to δ is equal to zero. That yields

$$(\mathbf{J}^T \mathbf{J})\delta = \mathbf{J}^T[\mathbf{y} - \mathbf{f}(\beta)]$$

where \mathbf{J} is the Jacobian matrix whose i^{th} row equals \mathbf{J}_i , and where \mathbf{f} and \mathbf{y} are vectors with i^{th} component $f(x_i, \beta)$ and y_i , respectively. Levenberg's contribution is to replace this equation by a “damped version”,

$$(\mathbf{J}^T \mathbf{J} + \lambda \mathbf{I})\delta = \mathbf{J}^T[\mathbf{y} - \mathbf{f}(\beta)] \quad (\text{C.3.4})$$

where \mathbf{I} is the identity matrix, giving as the increment, δ , to β .

Levenberg's algorithm has the disadvantage that if the value of damping factor, λ , is large, inverting $(\mathbf{J}^T \mathbf{J} + \lambda \mathbf{I})$ is not used at all. Therefore, Marquardt replaced \mathbf{I} in (C.3.4), with the diagonal matrix consisting of the diagonal elements of $\mathbf{J}^T \mathbf{J}$, resulting in the Levenberg-Marquardt algorithm:

$$(\mathbf{J}^T \mathbf{J} + \lambda \text{diag}(\mathbf{J}^T \mathbf{J}))\delta = \mathbf{J}^T[\mathbf{y} - \mathbf{f}(\beta)]. \quad (\text{C.3.5})$$

Appendix D

Data

D.1 Interest rate data

The market data presented here can be found in ([Ametrano and Bianchetti, 2013](#)).

| Instrument | Quote (ask, %) | Underlying | Start Date | Maturity Date | Settlement rule | Business Day Convention | End of Month Convention |
|------------|-------------------|------------|-----------------|-----------------|--------------------|----------------------------|----------------------------|
| Depo ON | 0.040 | Euribor1D | Tue 11 Dec 2012 | Wed 12 Dec 2012 | Today | Following | False |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Depo 12M | 0.540 | Euribor12M | Thu 13 Dec 2012 | Fri 13 Dec 2013 | Spot | Mod.Following | True |

Table D.1: EUR Deposit strip. Source: Reuters page KLIEM, 11 Dec. 2012.

| Instrument | Quote (bid, %) | Quote (ask, %) | Quote (mid, %) | Underlying | Start Date | Maturity |
|------------|-------------------|-------------------|-------------------|------------|-----------------|-----------------|
| FRA Tod3M | 0.156 | 0.206 | 0.181 | Euribor3M | Thu 13 Dec 2012 | Wed 13 Mar 2013 |
| FRA Tom3M | 0.154 | 0.204 | 0.179 | Euribor3M | Fri 14 Dec 2012 | Thu 14 Mar 2013 |
| FRA 1x4 | 0.140 | 0.190 | 0.165 | Euribor3M | Mon 14 Jan 2013 | Mon 15 Apr 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| FRA Tod6M | 0.291 | 0.341 | 0.316 | Euribor6M | Thu 13 Dec 2012 | Thu 13 Jun 2013 |
| FRA Tom6M | 0.287 | 0.337 | 0.312 | Euribor6M | Fri 14 Dec 2012 | Fri 14 Jun 2013 |
| FRA 1x7 | 0.268 | 0.318 | 0.293 | Euribor6M | Mon 14 Jan 2013 | Mon 15 Jul 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| FRA 12x24 | 0.482 | 0.532 | 0.507 | Euribor12M | Fri 13 Dec 2013 | Mon 15 Dec 2014 |
| FRA IMMF3 | 0.264 | 0.314 | 99.7110 | Euribor6M | Wed 16 Jan 2013 | Tue 16 Jul 2013 |
| FRA IMMG3 | 0.244 | 0.294 | 99.7310 | Euribor6M | Wed 20 Feb 2013 | Tue 20 Aug 2013 |
| FRA IMM3 | 0.234 | 0.284 | 99.7410 | Euribor6M | Wed 20 Mar 2013 | Fri 20 Sep 2013 |
| FRA IMM3 | 0.230 | 0.280 | 99.7450 | Euribor6M | Wed 17 Apr 2013 | Thu 17 Oct 2013 |

Table D.2: EUR FRA strips on Euribor 3M, Euribor 6M, and Euribor 12M.

| HW parameter | Value |
|----------------|---------|
| Mean reversion | 3% |
| Volatility | 0.3526% |

Table D.3: Hull-White parameters values for Futures 3M convexity adjustment as of 11 Dec. 2012.

| Instrument | Quote (bid, %) | Quote (ask, %) | Quote (mid, %) | Convexity adjustment | Underlying | Underlying Start Date | Underlying End Date |
|------------|-------------------|-------------------|-------------------|-------------------------|------------|--------------------------|------------------------|
| FUT 3MZ2 | 99.8200 | 99.8250 | 99.8225 | 0.0000% | Euribor3M | Wed 19 Dec 2012 | Tue 19 Mar 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮% | ⋮ | ⋮ | ⋮ |
| FUT 3MU7 | 98.4000 | 98.4250 | 98.4125 | 0.0136% | Euribor3M | Wed 20 Sep 2017 | Wed 20 Dec 2017 |

Table D.4: EUR Futures on Euribor 3M, "H", "M", "U" and "Z" standing for March, June, September and December expiries, respectively and two serial Futures (denoted as F3, G3 for Jan. and Feb 2013 expiries) are displayed. Source: Reuters page 0# FEI, as of 11 Dec. 2012. Column 5 reports the corresponding convexity adjustments.

| Instrument | Quote (bid, %) | Quote (ask, %) | Quote (mid, %) | Underlying | Start Date | Maturity |
|------------|-------------------|-------------------|-------------------|------------|-----------------|-----------------|
| AB6E1Y | 0.266 | 0.306 | 0.286 | Euribor6M | Thu 13 Dec 2012 | Fri 13 Dec 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| AB6E60Y | 2.443 | 2.483 | 2.463 | Euribor6M | Thu 13 Dec 2012 | Tue 18 Dec 2072 |

Table D.5: EUR IRS on Euribor 6M. The codes “AB6En” in col.1 label IRS receiving annually a fixed rate and paying semi-annually a floating rate on Euribor 6M with maturity in n months/years.

| Instrument | Quote (bid, %) | Quote (ask, %) | Quote (mid, %) | Underlying | Start Date | Maturity |
|------------|-------------------|-------------------|-------------------|------------|-----------------|-----------------|
| AB3E1Y | 0.116 | 0.166 | 0.141 | Euribor3M | Thu 13 Dec 2012 | Fri 13 Dec 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| AB3E50Y | 2.342 | 2.392 | 2.367 | Euribor3M | Thu 13 Dec 2012 | Wed 13 Dec 2062 |
| AB3EZ2 | 0.113 | 0.163 | 0.138 | Euribor3M | Wed 19 Dec 2012 | Thu 19 Dec 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| AB3EZ4 | 0.258 | 0.308 | 0.283 | Euribor3M | Wed 17 Jun 2014 | Mon 18 Jun 2017 |

Table D.6: EUR IRS on Euribor 3M, receiving annually a fixed rate and paying quarterly a floating rate on Euribor 3M. At the bottom IMM starting IRS are reported, denoted with their corresponding IMM code.

| Instrument | Quote (bid, %) | Quote (ask, %) | Quote (mid, %) | Underlying | Start Date | Maturity |
|------------|-------------------|-------------------|-------------------|------------|-----------------|-----------------|
| AB1E2M | 0.081 | 0.131 | 0.106 | Euribor1M | Thu 13 Dec 2012 | Wed 13 Feb 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| AB1E12M | 0.038 | 0.088 | 0.063 | Euribor1M | Thu 13 Dec 2012 | Fri 13 Dec 2013 |

Table D.7: EUR IRS on Euribor 1M, receiving annually a fixed rate and paying monthly a floating rate on Euribor 1M.

| Instrument | Quote (bid, %) | Quote (ask, %) | Quote (mid, %) | Underlying | Start Date | Maturity |
|------------|-------------------|-------------------|-------------------|------------|-----------------|-----------------|
| EON1W | 0.020 | 0.120 | 0.070 | Eonia | Thu 13 Dec 2012 | Thu 20 Dec 2012 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| EON30Y | 2.003 | 2.073 | 2.038 | Eonia | Thu 13 Dec 2012 | Mon 15 Dec 2042 |
| EONCBFEB13 | 0.021 | 0.071 | 0.046 | Eonia | Wed 16 Jan 2013 | Wed 13 Feb 2013 |
| EONCBMAR13 | -0.009 | 0.041 | 0.016 | Eonia | Wed 13 Feb 2013 | Wed 13 Mar 2013 |
| EONCBAPR13 | -0.032 | 0.018 | -0.007 | Eonia | Wed 13 Mar 2013 | Wed 10 Apr 2013 |
| EONCBMAY13 | -0.038 | 0.012 | -0.013 | Eonia | Wed 10 Apr 2013 | Wed 08 May 2013 |
| EONCBJUN13 | -0.039 | 0.011 | -0.014 | Eonia | Wed 08 May 2013 | Wed 12 Jun 2013 |
| EONCBJUL13 | -0.041 | 0.009 | -0.016 | Eonia | Wed 12 Jun 2013 | Wed 10 Jul 2013 |

Table D.8: EUR OIS, receiving an annual fixed rate and paying an annual floating rate on Eonia. At the bottom forward starting OIS at known ECB dates. Negatives OIS rates are enlightened in blue color.

| Instrument | Quote(bps) | Underlying 1st leg | Underlying 2nd leg | Start Date | Maturity |
|------------|------------|--------------------|--------------------|-----------------|-----------------|
| 1E6E1Y | 22.20 | Euribor1M | Euribor6M | Thu 13 Dec 2012 | Fri 13 Dec 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 1E6E30Y | 16.30 | Euribor1M | Euribor6M | Thu 13 Dec 2012 | Mon 15 Dec 2042 |
| 3E6E1Y | 14.50 | Euribor3M | Euribor6M | Thu 13 Dec 2012 | Fri 13 Dec 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 3E6E50Y | 5.40 | Euribor3M | Euribor6M | Thu 13 Dec 2012 | Wed 13 Dec 2062 |
| 6E12E1Y | 26.20 | Euribor6M | Euribor12M | Thu 13 Dec 2012 | Fri 13 Dec 2013 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 6E12E30Y | 6.60 | Euribor6M | Euribor12M | Thu 13 Dec 2012 | Mon 15 Dec 2042 |

Table D.9: EUR IRBS. The codes “ $xEyEnY$ ” in col. 1 label basis swaps receiving Euribor x M and paying Euribor y M plus basis spread with n maturity years.

D.2 Swedish market (swaption) data

| | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
|-----|------|------|------|------|------|------|------|------|------|------|------|
| 1Y | 33.4 | 30.1 | 31.5 | 31.1 | 34.3 | 33.9 | 29.8 | 27.8 | 24.5 | 22.4 | 19.7 |
| 2Y | 32.8 | 31.1 | 33.3 | 33.3 | 34 | 31.5 | 28.4 | 26.3 | 24.8 | 21.5 | 20.9 |
| 3Y | 34 | 32.5 | 34.8 | 34.8 | 32 | 29.9 | 27.4 | 25.4 | 24.4 | 21.1 | 21.5 |
| 4Y | 36.3 | 35 | 34.9 | 33.9 | 30.6 | 28.7 | 26.6 | 24.9 | 23.8 | 20.9 | 22.1 |
| 5Y | 37.4 | 36.3 | 35.5 | 33.1 | 29.7 | 27.6 | 25.9 | 24.4 | 23.2 | 20.8 | 22.2 |
| 7Y | 32.5 | 31.3 | 30.9 | 29.7 | 27.8 | 26.7 | 25.5 | 24.4 | 22.9 | 20.8 | 21.2 |
| 10Y | 31.1 | 29.3 | 28.2 | 26.8 | 26.1 | 25.7 | 25.1 | 24.4 | 22.9 | 21.1 | 19.9 |
| 12Y | 31.1 | 29.3 | 28.2 | 26.8 | 26.1 | 25.7 | 25.1 | 24.4 | 22.9 | 21.1 | 19.9 |
| 15Y | 29.3 | 28 | 27.3 | 26.1 | 25.1 | 24.8 | 24.1 | 23 | 22.3 | 20.3 | 17.2 |
| 20Y | 28.9 | 27.9 | 27.4 | 26.4 | 25 | 24.6 | 23.8 | 22.8 | 21.9 | 19.8 | 16.1 |

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| 1Y | | | | | | | | | | | | |
|-----------|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| -0.568949 | -2 | 59.9 | 59.9 | 45.4 | 26.3 | 24.8 | 17 | 13.98769 | 11 | 9.1 | 8.4 | 6.8 |
| -0.318949 | -1.75 | 51.25 | 51.25 | 38.9 | 22.6 | 20.175 | 13.975 | 11.51495 | 9.075 | 7.5 | 6.925 | 5.625 |
| -0.068949 | -1.5 | 42.6 | 42.6 | 32.4 | 18.9 | 15.55 | 10.95 | 9.042202 | 7.15 | 5.9 | 5.45 | 4.45 |
| 0.181051 | -1.25 | 33.95 | 33.95 | 25.9 | 15.2 | 10.925 | 7.925 | 6.56946 | 5.225 | 4.3 | 3.975 | 3.275 |
| 0.431051 | -1 | 25.3 | 25.3 | 19.4 | 11.5 | 6.3 | 4.9 | 4.096717 | 3.3 | 2.7 | 2.5 | 2.1 |
| 0.681051 | -0.75 | 16.65 | 16.65 | 12.9 | 7.8 | 4.35 | 3.4 | 2.847743 | 2.3 | 1.85 | 1.7 | 1.45 |
| 0.931051 | -0.5 | 8 | 8 | 6.4 | 4.1 | 2.4 | 1.9 | 1.598769 | 1.3 | 1 | 0.9 | 0.8 |
| 1.181051 | -0.25 | 3.1 | 3.1 | 2.6 | 1.7 | 1.1 | 0.9 | 0.749384 | 0.6 | 0.4 | 0.4 | 0.4 |
| 1.431051 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1.681051 | 0.25 | -1.4 | -1.4 | -1.4 | -1.1 | -0.7 | -0.6 | -0.49959 | -0.4 | -0.4 | -0.3 | -0.3 |
| 1.931051 | 0.5 | -1.6 | -1.6 | -1.9 | -1.8 | -1.3 | -1.2 | -0.948974 | -0.7 | -0.6 | -0.5 | -0.6 |
| 2.181051 | 0.75 | -1.05 | -1.05 | -1.8 | -2.1 | -1.7 | -1.5 | -1.198769 | -0.9 | -0.7 | -0.55 | -0.75 |
| 2.431051 | 1 | -0.5 | -0.5 | -1.7 | -2.4 | -2.1 | -1.8 | -1.448564 | -1.1 | -0.8 | -0.6 | -0.9 |
| 2.681051 | 1.25 | 0.35 | 0.35 | -1.15 | -2.25 | -2.2 | -1.925 | -1.523358 | -1.125 | -0.775 | -0.55 | -0.975 |
| 2.931051 | 1.5 | 1.2 | 1.2 | -0.6 | -2.1 | -2.3 | -2.05 | -1.598153 | -1.15 | -0.75 | -0.5 | -1.05 |
| 3.181051 | 1.75 | 2.05 | 2.05 | -0.05 | -1.95 | -2.4 | -2.175 | -1.672948 | -1.175 | -0.725 | -0.45 | -1.125 |
| 3.431051 | 2 | 2.9 | 2.9 | 0.5 | -1.8 | -2.5 | -2.3 | -1.747743 | -1.2 | -0.7 | -0.4 | -1.2 |
| 2Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| -0.280702 | -2 | 40.5 | 40.5 | 31.6 | 40.8 | 19.5 | 13.9 | 11.84159 | 9.8 | 8.5 | 7.3 | 6.6 |
| -0.030702 | -1.75 | 34.7 | 34.7 | 27.1 | 32.75 | 15.925 | 11.425 | 9.743126 | 8.075 | 6.975 | 5.95 | 5.425 |
| 0.219298 | -1.5 | 28.9 | 28.9 | 22.6 | 24.7 | 12.35 | 8.95 | 7.644665 | 6.35 | 5.45 | 4.6 | 4.25 |
| 0.469298 | -1.25 | 23.1 | 23.1 | 18.1 | 16.65 | 8.775 | 6.475 | 5.546204 | 4.625 | 3.925 | 3.25 | 3.075 |
| 0.719298 | -1 | 17.3 | 17.3 | 13.6 | 8.6 | 5.2 | 4 | 3.447743 | 2.9 | 2.4 | 1.9 | 1.9 |
| 0.969298 | -0.75 | 11.5 | 11.5 | 9.1 | 5.85 | 3.55 | 2.75 | 2.373461 | 2 | 1.65 | 1.3 | 1.35 |
| 1.219298 | -0.5 | 5.7 | 5.7 | 4.6 | 3.1 | 1.9 | 1.5 | 1.299179 | 1.1 | 0.9 | 0.7 | 0.8 |
| 1.469298 | -0.25 | 2.2 | 2.2 | 1.9 | 1.3 | 0.8 | 0.6 | 0.549795 | 0.5 | 0.4 | 0.3 | 0.3 |
| 1.719298 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1.969298 | 0.25 | -1.1 | -1.1 | -1.1 | -0.9 | -0.7 | -0.5 | -0.39959 | -0.3 | -0.2 | -0.1 | -0.3 |
| 2.219298 | 0.5 | -1.3 | -1.3 | -1.6 | -1.5 | -1.1 | -0.9 | -0.699179 | -0.5 | -0.4 | -0.2 | -0.5 |
| 2.469298 | 0.75 | -0.85 | -0.85 | -1.55 | -1.75 | -1.4 | -1.15 | -0.873871 | -0.6 | -0.45 | -0.15 | -0.65 |
| 2.719298 | 1 | -0.4 | -0.4 | -1.5 | -2 | -1.7 | -1.4 | -1.048564 | -0.7 | -0.5 | -0.1 | -0.8 |
| 2.969298 | 1.25 | 0.35 | 0.35 | -1.05 | -1.925 | -1.775 | -1.425 | -1.03591 | -0.65 | -0.425 | 0.025 | -0.85 |
| 3.219298 | 1.5 | 1.1 | 1.1 | -0.6 | -1.85 | -1.85 | -1.45 | -1.023256 | -0.6 | -0.35 | 0.15 | -0.9 |
| 3.469298 | 1.75 | 1.85 | 1.85 | -0.15 | -1.775 | -1.925 | -1.475 | -1.010602 | -0.55 | -0.275 | 0.275 | -0.95 |
| 3.719298 | 2 | 2.6 | 2.6 | 0.3 | -1.7 | -2 | -1.5 | -0.997948 | -0.5 | -0.2 | 0.4 | -1 |
| 3Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| -0.005595 | -2 | 38.79689 | 38.79689 | 30.1 | 32.86667 | 16.80246 | 12.50128 | 10.90847 | 9.333333 | 8.233577 | 7.200091 | 7.034124 |
| 0.244405 | -1.75 | 32.6713 | 32.6713 | 25.35833 | 26.40833 | 13.70203 | 10.25107 | 8.951591 | 7.666667 | 6.74188 | 5.858417 | 5.783987 |
| 0.494405 | -1.5 | 26.54571 | 26.54571 | 20.61667 | 19.95 | 10.6016 | 8.000867 | 6.994712 | 6 | 5.250182 | 4.516743 | 4.53385 |
| 0.744405 | -1.25 | 20.42012 | 20.42012 | 15.875 | 13.49167 | 7.501163 | 5.750661 | 5.037834 | 4.333333 | 3.758485 | 3.175068 | 3.283714 |
| 0.994405 | -1 | 14.29453 | 14.29453 | 11.13333 | 7.033333 | 4.40073 | 3.500456 | 3.080955 | 2.666667 | 2.266788 | 1.833394 | 2.033577 |
| 1.244405 | -0.75 | 9.479653 | 9.479653 | 7.433333 | 4.766667 | 2.98385 | 2.383668 | 2.106719 | 1.833333 | 1.53344 | 1.233394 | 1.433485 |
| 1.494405 | -0.5 | 4.664781 | 4.664781 | 3.733333 | 2.5 | 1.566971 | 1.26688 | 1.132483 | 1 | 0.800091 | 0.633394 | 0.833394 |
| 1.744405 | -0.25 | 1.79927 | 1.79927 | 1.5 | 1.033333 | 0.666788 | 0.500091 | 0.466378 | 0.433333 | 0.333394 | 0.266697 | 0.333394 |
| 1.994405 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2.244405 | 0.25 | -0.866241 | -0.866241 | -0.866667 | -0.7 | -0.566788 | -0.400091 | -0.332938 | -0.266667 | -0.2 | -0.133303 | -0.3 |
| 2.494405 | 0.5 | -1.032847 | -1.032847 | -1.2 | -1.133333 | -0.86688 | -0.700182 | -0.549111 | -0.4 | -0.366697 | -0.2 | -0.533394 |
| 2.744405 | 0.75 | -0.616241 | -0.616241 | -1.05 | -1.283333 | -1.083622 | -0.883577 | -0.673781 | -0.466667 | -0.383394 | -0.15 | -0.700091 |
| 2.994405 | 1 | -0.199635 | -0.199635 | -0.9 | -1.433333 | -1.300365 | -1.066971 | -0.79845 | -0.533333 | -0.400091 | -0.1 | -0.866788 |
| 3.244405 | 1.25 | 0.508622 | 0.508622 | -0.483333 | -1.316667 | -1.317085 | -1.050342 | -0.689976 | -0.466667 | -0.316766 | 0.025 | -0.925137 |
| 3.494405 | 1.5 | 1.21688 | 1.21688 | -0.066667 | -1.2 | -1.333805 | -1.033714 | -0.681548 | -0.4 | -0.23344 | 0.15 | -0.983485 |
| 3.744405 | 1.75 | 1.925137 | 1.925137 | 0.35 | -1.083333 | -1.350525 | -1.017085 | -0.67312 | -0.333333 | -0.150114 | 0.275 | -1.041834 |
| 3.994405 | 2 | 2.633394 | 2.633394 | 0.766667 | -0.966667 | -1.367245 | -1.000456 | -0.664692 | -0.266667 | -0.066788 | 0.4 | -1.100182 |
| 4Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| 0.213859 | -2 | 37.09845 | 37.09845 | 28.6 | 24.93333 | 14.10493 | 11.09872 | 9.977901 | 8.866667 | 7.966423 | 7.100182 | 7.467062 |
| 0.463859 | -1.75 | 30.64815 | 30.64815 | 23.61667 | 20.06667 | 11.47906 | 9.073928 | 8.162218 | 7.258333 | 6.50812 | 5.766834 | 6.141994 |
| 0.713859 | -1.5 | 24.19786 | 24.19786 | 18.63333 | 15.2 | 8.853193 | 7.049133 | 6.346535 | 5.65 | 5.049818 | 4.433485 | 4.816925 |
| 0.963859 | -1.25 | 17.74756 | 17.74756 | 13.65 | 10.33333 | 6.227327 | 5.024339 | 4.530853 | 4.0416667 | 3.591515 | 3.100137 | 3.491857 |
| 1.213859 | -1 | 11.29726 | 11.29726 | 8.666667 | 5.466667 | 3.60146 | 2.999544 | 2.71517 | 2.433333 | 2.133212 | 1.766788 | 2.166788 |
| 1.463859 | -0.75 | 7.464827 | 7.464827 | 5.766667 | 3.683333 | 2.41771 | 2.016332 | 1.840706 | 1.666667 | 1.41656 | 1.166788 | 1.516743 |
| 1.713859 | -0.5 | 3.632391 | 3.632391 | 2.866667 | 1.9 | 1.233942 | 1.03312 | 0.966241 | 0.9 | 0.699909 | 0.566788 | 0.866697 |
| 1.963859 | -0.25 | 1.399635 | 1.399635 | 1.1 | 0.766667 | 0.533577 | 0.399909 | 0.383189 | 0.366667 | 0.266606 | 0.233394 | 0.366697 |
| 2.213859 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2.463859 | 0.25 | -0.63312 | -0.63312 | -0.633333 | -0.5 | -0.433577 | -0.299909 | -0.266469 | -0.233333 | -0.2 | -0.166606 | -0.3 |
| 2.713859 | 0.5 | -0.766423 | -0.766423 | -0.8 | -0.766667 | -0.633759 | -0.499818 | -0.399453 | -0.3 | -0.333303 | -0.2 | -0.566697 |
| 2.963859 | 0.75 | -0.38312 | -0.38312 | -0.55 | -0.816667 | -0.767245 | -0.616423 | -0.474236 | -0.333333 | -0.316606 | -0.15 | -0.750046 |
| 3.213859 | 1 | 0.000182 | 0.000182 | -0.3 | -0.866667 | -0.90073 | -0.733029 | -0.54902 | -0.366667 | -0.299909 | -0.1 | -0.933394 |
| 3.463859 | 1.25 | 0.666811 | 0.666811 | 0.083333 | -0.708333 | -0.85917 | -0.674658 | -0.344988 | -0.283333 | -0.208234 | 0.025 | -1.000068 |
| 3.713859 | 1.5 | 1.33344 | 1.33344 | 0.466667 | -0.55 | -0.817609 | -0.616286 | -0.340774 | -0.2 | -0.1165 | 0.15 | -1.066743 |
| 3.963859 | 1.75 | 2.000068 | 2.000068 | 0.85 | -0.391667 | -0.776049 | -0.557915 | -0.33656 | -0.116667 | -0.024886 | 0.275 | -1.133417 |
| 4.213859 | 2 | 2.666697 | 2.666697 | 1.233333 | -0.233333 | -0.734489 | -0.499544 | -0.332346 | -0.033333 | 0.066788 | 0.4 | -1.200091 |
| 5Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| 0.399829 | -2 | 35.4 | 35.4 | 27.1 | 17 | 11.4 | 9.7 | 9.047332 | 8.4 | 7.7 | 7 | 7.9 |
| 0.649829 | -1.75 | 28.625 | 28.625 | 21.875 | 13.725 | 9.25 | 7.9 | 7.372845 | 6.85 | 6.275 | 5.675 | 6.5 |
| 0.899829 | -1.5 | 21.85 | 21.85 | 16.65 | 10.45 | 7.1 | 6.1 | 5.698358 | 5.3 | 4.85 | 4.35 | 5.1 |
| 1.149829 | -1.25 | 15.075 | 15.075 | 11.425 | 7.175 | 4.95 | 4.3 | 4.023871 | 3.75 | 3.425 | 3.025 | 3.7 |
| 1.399829 | -1 | 8.3 | 8.3 | 6.2 | 3.9 | 2.8 | 2.5 | 2.349384 | 2.2 | 2 | 1.7 | 2.3 |
| 1.649829 | -0.75 | 5.45 | 5.45 | 4.1 | 2.6 | 1.85 | 1.65 | 1.574692 | 1.5 | 1.3 | 1.1 | 1.6 |
| 1.899829 | -0.5 | 2.6 | 2.6 | 2 | 1.3 | 0.9 | 0.8 | 0.8 | 0.8 | 0.6 | 0.5 | 0.9 |
| 2.149829 | -0.25 | 1 | 1 | 0.7 | 0.5 | 0.4 | 0.3 | 0.3 | 0.3 | 0.2 | 0.2 | 0.4 |
| 2.399829 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2.649829 | 0.25 | -0.4 | -0.4 | -0.4 | -0.3 | -0.3 | -0.2 | -0.2 | -0.2 | -0. | | |

APPENDIX D. DATA

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| 7Y | | | | | | | | | | | | |
|----------|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|------------|-----------|-----------|
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| 0.673166 | -2 | 31.03642 | 31.03642 | 24.13919 | 15.39956 | 11.12015 | 10.21972 | 10.07022 | 9.920416 | 9.259146 | 8.559146 | 8.620591 |
| 0.923166 | -1.75 | 25.24222 | 25.24222 | 19.54436 | 12.46466 | 9.040115 | 8.359748 | 8.260318 | 8.160359 | 7.624261 | 7.024261 | 7.120509 |
| 1.173166 | -1.5 | 19.44803 | 19.44803 | 14.94953 | 9.529748 | 6.960077 | 6.499781 | 6.450412 | 6.400301 | 5.989376 | 5.489376 | 5.620427 |
| 1.423166 | -1.25 | 13.65383 | 13.65383 | 10.35471 | 6.594841 | 4.880038 | 4.639814 | 4.640506 | 4.640244 | 4.354491 | 3.954491 | 4.120345 |
| 1.673166 | -1 | 7.859639 | 7.859639 | 5.75988 | 3.659934 | 2.8 | 2.779847 | 2.8306 | 2.880186 | 2.719606 | 2.419606 | 2.620263 |
| 1.923166 | -0.75 | 5.209803 | 5.209803 | 3.799918 | 2.43996 | 1.869989 | 1.86988 | 1.925514 | 1.980131 | 1.839704 | 1.619715 | 1.8201801 |
| 2.173166 | -0.5 | 2.559967 | 2.559967 | 1.839956 | 1.219978 | 0.939978 | 0.950912 | 1.020427 | 1.080077 | 0.959803 | 0.819825 | 1.020099 |
| 2.423166 | -0.25 | 1 | 1 | 0.659989 | 0.459989 | 0.4 | 0.379956 | 0.420263 | 0.460044 | 0.39989 | 0.359912 | 0.440033 |
| 2.673166 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2.923166 | 0.25 | -0.440033 | -0.440033 | -0.279967 | -0.3 | -0.260022 | -0.239978 | -0.28023 | -0.320033 | -0.279956 | -0.319934 | -0.380066 |
| 3.173166 | 0.5 | -0.580066 | -0.580066 | -0.159934 | -0.359989 | -0.360022 | -0.379956 | -0.430353 | -0.480077 | -0.49989 | -0.439869 | -0.720099 |
| 3.423166 | 0.75 | -0.270099 | -0.270099 | 0.17006 | -0.309989 | -0.390033 | -0.429956 | -0.485399 | -0.540093 | -0.529847 | -0.489814 | -0.960131 |
| 3.673166 | 1 | 0.039869 | 0.039869 | 0.500055 | -0.259989 | -0.420044 | -0.479956 | -0.540444 | -0.600109 | -0.559803 | -0.539759 | -1.200164 |
| 3.923166 | 1.25 | 0.644852 | 0.644852 | 0.950082 | -0.039984 | -0.300055 | -0.379956 | -0.33583 | -0.530118 | -0.489786 | -0.454737 | -1.305189 |
| 4.173166 | 1.5 | 1.249836 | 1.249836 | 1.400109 | 0.180022 | -0.180066 | -0.279956 | -0.310871 | -0.460126 | -0.41977 | -0.369715 | -1.410214 |
| 4.423166 | 1.75 | 1.854819 | 1.854819 | 1.850137 | 0.400027 | -0.060077 | -0.179956 | -0.285912 | -0.390134 | -0.349754 | -0.284693 | -1.515238 |
| 4.673166 | 2 | 2.459803 | 2.459803 | 2.300164 | 0.620033 | 0.059912 | -0.079956 | -0.260953 | -0.320142 | -0.279737 | -0.199671 | -1.620263 |
| 10Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| 0.905638 | -2 | 24.5 | 24.5 | 19.7 | 13 | 10.7 | 11 | 11.60246 | 12.2 | 11.6 | 10.9 | 9.7 |
| 1.155638 | -1.75 | 20.175 | 20.175 | 16.05 | 10.575 | 8.725 | 9.05 | 9.589706 | 10.125 | 9.65 | 9.05 | 8.05 |
| 1.405638 | -1.5 | 15.85 | 15.85 | 12.4 | 8.15 | 6.75 | 7.1 | 7.576949 | 8.05 | 7.7 | 7.2 | 6.4 |
| 1.655638 | -1.25 | 11.525 | 11.525 | 8.75 | 5.725 | 4.775 | 5.15 | 5.564193 | 5.975 | 5.75 | 5.35 | 4.75 |
| 1.905638 | -1 | 7.2 | 7.2 | 5.1 | 3.3 | 2.8 | 3.2 | 3.551436 | 3.9 | 3.8 | 3.5 | 3.1 |
| 2.155638 | -0.75 | 4.85 | 4.85 | 3.35 | 2.2 | 1.9 | 2.2 | 2.451026 | 2.7 | 2.65 | 2.4 | 2.15 |
| 2.405638 | -0.5 | 2.5 | 2.5 | 1.6 | 1.1 | 1 | 1.2 | 1.350616 | 1.5 | 1.5 | 1.3 | 1.2 |
| 2.655638 | -0.25 | 1 | 1 | 0.6 | 0.4 | 0.4 | 0.5 | 0.60041 | 0.7 | 0.7 | 0.6 | 0.5 |
| 2.905638 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.155638 | 0.25 | -0.5 | -0.5 | -0.1 | -0.3 | -0.2 | -0.3 | -0.40041 | -0.5 | -0.4 | -0.5 | -0.5 |
| 3.405638 | 0.5 | -0.7 | -0.7 | 0.2 | -0.3 | -0.3 | -0.5 | -0.700821 | -0.9 | -0.8 | -0.8 | -0.9 |
| 3.655638 | 0.75 | -0.45 | -0.45 | 0.5 | -0.25 | -0.3 | -0.55 | -0.801026 | -1.05 | -0.95 | -1 | -1.2 |
| 3.905638 | 1 | -0.2 | -0.2 | 0.8 | -0.2 | -0.3 | -0.6 | -0.901231 | -1.2 | -1.1 | -1.2 | -1.5 |
| 4.155638 | 1.25 | 0.375 | 0.375 | 1.4 | 0.05 | -0.15 | -0.5 | -0.838885 | -1.175 | -1.075 | -1.175 | -1.65 |
| 4.405638 | 1.5 | 0.95 | 0.95 | 2 | 0.3 | 0 | -0.4 | -0.776539 | -1.15 | -1.05 | -1.15 | -1.8 |
| 4.655638 | 1.75 | 1.525 | 1.525 | 2.6 | 0.55 | 0.15 | -0.3 | -0.714193 | -1.125 | -1.025 | -1.125 | -1.95 |
| 4.905638 | 2 | 2.1 | 2.1 | 3.2 | 0.8 | 0.3 | -0.2 | -0.651847 | -1.1 | -1 | -1.1 | -2.1 |
| 12Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| 1.000762 | -2 | 24.01974 | 24.01974 | 19.26024 | 12.92004 | 10.26024 | 10.93997 | 11.58262 | 12.21999 | 11.68004 | 10.97996 | 9.339803 |
| 1.250762 | -1.75 | 19.79479 | 19.79479 | 15.6852 | 10.51503 | 8.315225 | 8.999973 | 9.57233 | 10.13999 | 9.710033 | 9.114964 | 7.754838 |
| 1.500762 | -1.5 | 15.56985 | 15.56985 | 12.11016 | 8.110022 | 6.370208 | 7.059978 | 7.562044 | 8.059995 | 7.740022 | 7.249973 | 6.169874 |
| 1.750762 | -1.25 | 11.3449 | 11.3449 | 8.535118 | 5.705011 | 4.425192 | 5.119984 | 5.551758 | 5.979997 | 5.770011 | 5.384981 | 4.58491 |
| 2.000762 | -1 | 7.119956 | 7.119956 | 4.960077 | 3.3 | 2.480175 | 3.179989 | 3.541472 | 3.9 | 3.8 | 3.519989 | 2.999945 |
| 2.250762 | -0.75 | 4.799973 | 4.799973 | 3.220071 | 2.190005 | 1.59017 | 2.179989 | 2.441062 | 2.7 | 2.65 | 2.419989 | 2.089967 |
| 2.500762 | -0.5 | 2.479989 | 2.479989 | 1.480066 | 1.080011 | 0.700164 | 1.179989 | 1.340651 | 1.5 | 1.5 | 1.319989 | 1.179989 |
| 2.750762 | -0.25 | 0.979989 | 0.979989 | 0.580011 | 0.4 | 0.120153 | 0.5 | 0.60041 | 0.7 | 0.7 | 0.6 | 0.5 |
| 3.000762 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.250762 | 0.25 | -0.5 | -0.5 | -0.199945 | -0.280011 | -0.479847 | -0.3 | -0.40041 | -0.5 | -0.4 | -0.480011 | -0.459978 |
| 3.500762 | 0.5 | -0.659978 | -0.659978 | 0.080066 | -0.280011 | -0.579847 | -0.479989 | -0.68081 | -0.880011 | -0.779989 | -0.780011 | -0.839967 |
| 3.750762 | 0.75 | -0.399973 | -0.399973 | 0.420044 | -0.210022 | -0.569852 | -0.529989 | -0.781015 | -1.030011 | -0.929989 | -0.970016 | -1.119956 |
| 4.000762 | 1 | -0.139967 | -0.139967 | 0.760022 | -0.140033 | -0.559858 | -0.579989 | -0.88122 | -1.180011 | -1.079989 | -1.160022 | -1.399945 |
| 4.250762 | 1.25 | 0.445038 | 0.445038 | 1.365019 | 0.124959 | -0.409858 | -0.469984 | -0.671016 | -1.145016 | -1.049986 | -1.130025 | -1.529934 |
| 4.500762 | 1.5 | 1.030044 | 1.030044 | 1.970016 | 0.389951 | -0.259858 | -0.359978 | -0.621146 | -1.110022 | -1.0199834 | -1.100027 | -1.659923 |
| 4.750762 | 1.75 | 1.615049 | 1.615049 | 2.575014 | 0.654942 | -0.109858 | -0.249973 | -0.571276 | -1.075027 | -0.989981 | -1.07003 | -1.789912 |
| 5.000762 | 2 | 2.200055 | 2.200055 | 3.180011 | 0.919934 | 0.040142 | -0.139967 | -0.521406 | -1.040033 | -0.959978 | -1.040033 | -1.919901 |
| 15Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| 1.104474 | -2 | 23.29967 | 23.29967 | 18.6 | 12.8 | 9.6 | 10.85004 | 11.55286 | 12.25 | 11.79995 | 11.1 | 8.799754 |
| 1.354474 | -1.75 | 19.22474 | 19.22474 | 15.1375 | 10.425 | 7.7 | 8.925034 | 9.546277 | 10.1625 | 9.799959 | 9.2125 | 7.312298 |
| 1.604474 | -1.5 | 15.14981 | 15.14981 | 11.675 | 8.05 | 5.8 | 7.000027 | 7.539696 | 8.075 | 7.799973 | 7.325 | 5.824843 |
| 1.854474 | -1.25 | 11.07488 | 11.07488 | 8.2125 | 5.675 | 3.9 | 5.075021 | 5.533114 | 5.9875 | 5.799986 | 5.4375 | 4.337387 |
| 2.104474 | -1 | 6.999945 | 6.999945 | 4.75 | 3.3 | 2 | 3.150014 | 3.526532 | 3.9 | 3.8 | 3.55 | 2.849932 |
| 2.354474 | -0.75 | 4.724966 | 4.724966 | 3.025 | 2.175 | 1.125 | 2.150014 | 2.426122 | 2.7 | 2.65 | 2.45 | 1.999959 |
| 2.604474 | -0.5 | 2.449986 | 2.449986 | 1.3 | 1.05 | 0.25 | 1.150014 | 1.325711 | 1.5 | 1.5 | 1.35 | 1.149986 |
| 2.854474 | -0.25 | 0.949986 | 0.949986 | 0.55 | 0.4 | -0.3 | 0.5 | 0.60041 | 0.7 | 0.7 | 0.6 | 0.5 |
| 3.104474 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.354474 | 0.25 | -0.5 | -0.5 | -0.35 | -0.25 | -0.9 | -0.3 | -0.40041 | -0.5 | -0.4 | -0.45 | -0.399973 |
| 3.604474 | 0.5 | -0.599973 | -0.599973 | -0.1 | -0.25 | -1 | -0.450014 | -0.650807 | -0.85 | -0.750014 | -0.75 | -0.749959 |
| 3.854474 | 0.75 | -0.324966 | -0.324966 | 0.3 | -0.15 | -0.975 | -0.500014 | -0.751012 | -1 | -0.900014 | -0.925 | -0.999945 |
| 4.104474 | 1 | -0.049959 | -0.049959 | 0.7 | -0.05 | -0.95 | -0.550014 | -0.851218 | -1.15 | -1.050014 | -1.1 | -1.249932 |
| 4.354474 | 1.25 | 0.550048 | 0.550048 | 1.3125 | 0.2375 | -0.8 | -0.425021 | -0.419328 | -1.1 | -1.012517 | -1.0625 | -1.349918 |
| 4.604474 | 1.5 | 1.150055 | 1.150055 | 1.925 | 0.525 | -0.65 | -0.300027 | -0.388163 | -1.05 | -0.975021 | -1.025 | -1.449904 |
| 4.854474 | 1.75 | 1.750062 | 1.750062 | 2.5375 | 0.8125 | -0.5 | -0.175034 | -0.356999 | -1 | -0.937524 | -0.9875 | -1.549891 |
| 5.104474 | 2 | 2.350068 | 2.350068 | 3.15 | 1.1 | -0.35 | -0.050041 | -0.325834 | -0.95 | -0.900027 | -0.95 | -1.649877 |
| 20Y | | | | | | | | | | | | |
| Strike | Spread | 1M | 3M | 6M | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y | 10Y | 20Y |
| 1.166501 | -2 | 22.1 | 22.1 | 17.5 | 12.6 | 8.5 | 10.7 | 11.50328 | 12.3 | 12 | 11.3 | 7.9 |
| 1.416501 | -1.75 | 18.275 | 18.275 | 14.225 | 10.275 | 6.675 | 8.8 | 9.502873 | 10.2 | 9.95 | 9.375 | 6.575 |
| 1.666501 | -1.5 | 14.45 | 14.45 | 10.95 | 7.95 | 4.85 | 6.9 | 7.502462 | 8.1 | 7.9 | 7.45 | 5.25 |
| 1.916501 | -1.25 | 10.625 | 10.625 | 7.675 | 5.625 | 3.025 | 5 | 5.502052 | 6 | 5.85 | 5.525 | 3.925 |
| 2.166501 | -1 | 6.8 | 6.8 | 4.4 | 3.3 | 1.2 | 3.1 | 3.501642 | 3.9 | 3.8 | 3.6 | 2.6 |
| 2.416501 | -0.75 | 4.6 | 4.6 | 2.7 | 2.15 | 0.35 | 2.1 | 2.401231 | 2.7 | 2.65 | 2.5 | 1.85 |
| 2.666501 | -0.5 | 2.4 | 2.4 | 1 | 1 | -0.5 | 1.1 | 1.300821 | 1.5 | 1.5 | 1.4 | 1.1 |
| 2.916501 | -0.25 | 0.9 | 0.9 | 0.5 | 0.4 | -1 | 0.5 | 0.60041 | 0.7 | 0.7 | 0.6 | 0.5 |
| 3.166501 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3.416501 | 0.25 | -0.5 | -0.5 | -0.6 | -0.2 | -1.6 | -0.3 | -0.40041 | -0.5 | -0.4 | -0.4 | -0. |

Appendix E

Interest rate derivatives pricing formulas

We derive the pricing formulas for FRA, Futures, IRS, OIS and IRBS used in Chapters 2 and 3.

E.1 FRA

The payoff of the market FRA is given in equation (3.3.5). The price of the market FRA at time $t < T_{j-1}$ under the measure $\mathbb{Q}_c^{T_{j-1}}$, using equation (A.0.3), is given by

$$\begin{aligned} \mathbf{FRA}_{\text{Mkt}}(t; \mathbf{T}, K, \omega) &= P_c(t; T_{j-1}) \mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left[\mathbf{FRA}_{\text{Mkt}}(T_{j-1}; \mathbf{T}, K, \omega) \right] \\ &= N\omega P_c(t; T_{j-1}) \mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left\{ \frac{[L_x(T_{j-1}, T_j) - K] \delta_L(T_{j-1}, T_j)}{1 + L_x(T_{j-1}, T_j) \delta_L(T_{j-1}, T_j)} \right\} \\ &= N\omega P_c(t; T_{j-1}) \left[1 - (1 + K \delta_L(T_{j-1}, T_j)) \mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left\{ \frac{1}{1 + L_x(T_{j-1}, T_j) \delta_L(T_{j-1}, T_j)} \right\} \right]. \end{aligned} \quad (\text{E.1.1})$$

To pass from $\mathbb{Q}_c^{T_{j-1}}$ to $\mathbb{Q}_c^{T_j}$, we let A be a price of an asset, using Proposition 2.6, we have

$$\frac{A(t, T_j)}{P_c(t; T_{j-1})} = \mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left[\frac{A(T_{j-1}, T_j)}{P_c(T_{j-1}; T_{j-1})} \right] \quad \text{and} \quad \frac{A(t, T_j)}{P_c(t; T_j)} = \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\frac{A(T_{j-1}, T_j)}{P_c(T_{j-1}; T_j)} \right].$$

Hence

$$\mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left[\frac{1}{1 + L_x(T_{j-1}, T_j) \delta_L(T_{j-1}, T_j)} \right] = \frac{1}{1 + F_{c,j}(t) \delta_c(T_{j-1}, T_j)} \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\frac{1 + L_c(T_{j-1}, T_j) \delta_c(T_{j-1}, T_j)}{1 + L_x(T_{j-1}, T_j) \delta_L(T_{j-1}, T_j)} \right].$$

Therefore, we have

$$\mathbf{FRA}_{\text{Mkt}}(t; \mathbf{T}, K, \omega) = N\omega P_c(t; T_{j-1}) \left\{ 1 - \frac{1 + K \delta_L(T_{j-1}, T_j)}{1 + F_{c,j}(t) \delta_c(T_{j-1}, T_j)} \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\frac{1 + L_c(T_{j-1}, T_j) \delta_c(T_{j-1}, T_j)}{1 + L_x(T_{j-1}, T_j) \delta_L(T_{j-1}, T_j)} \right] \right\}.$$

The FRA equilibrium rate is the value of K that causes the contract to have zero value at time t , i.e. such that $\mathbf{FRA}_{Mkt}(t; \mathbf{T}, K, \omega) = 0$, we have

$$R_{x,Mkt}^{FRA}(t, T_{j-1}, T_j) = \frac{1}{\delta_L(T_{j-1}, T_j)} \left\{ \frac{1 + F_{c,j}(t)\delta_c(T_{j-1}, T_j)}{\mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\frac{1 + L_c(T_{j-1}, T_j)\delta_c(T_{j-1}, T_j)}{1 + L_x(T_{j-1}, T_j)\delta_x(T_{j-1}, T_j)} \right]} - 1 \right\} \quad (\text{E.1.2})$$

We can see that the price and the equilibrium rate of the market FRA depend on the expectation of the ration between the two rates $L_c(T_{j-1}, T_j)$ and $L_x(T_{j-1}, T_j)$ under the forward measure $\mathbb{Q}_c^{T_j}$. The expectation depend on the particular model chosen for the joint distribution of these two rates. We have, in general

$$\mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\frac{1 + L_c(T_{j-1}, T_j)\delta_c(T_{j-1}, T_j)}{1 + L_x(T_{j-1}, T_j)\delta_x(T_{j-1}, T_j)} \right] = \frac{1 + F_{c,j}(t)\delta_c(T_{j-1}, T_j)}{1 + F_{x,j}(t)\delta_x(T_{j-1}, T_j)} e^{C_{c,x}^{FRA}(t; T_{j-1})}. \quad (\text{E.1.3})$$

It follows that

$$\mathbf{FRA}_{Mkt}(t; \mathbf{T}, K, \omega) = N\omega P_c(t; T_{j-1}) \left[1 - \frac{1 + K\delta_x(T_{j-1}, T_j)}{1 + F_{x,j}(t)\delta_x(T_{j-1}, T_j)} e^{C_{c,x}^{FRA}(t; T_{j-1})} \right] \quad (\text{E.1.4})$$

and

$$R_{x,Mkt}^{FRA}(t, \mathbf{T}) = \frac{1}{\delta_x(T_{j-1}, T_j)} \left\{ \left[1 + F_{x,j}(t)\delta_x(T_{j-1}, T_j) \right] e^{-C_{c,x}^{FRA}(t; T_{j-1})} - 1 \right\}, \quad (\text{E.1.5})$$

with $e^{C_{c,x}^{FRA}(t; T_{j-1})}$ being the convexity adjustment, whose explicit expression depends on the chosen model for the dynamics of $F_{c,j}(t)$ and $F_{x,j}(t)$. Its actual size i.e. post credit crunch is below 1bp, even for long maturity.

E.2 Futures

The payoff of Futures is given in equation (3.3.12). The price of Futures at time $t < T_{j-1}$ under the measure $\mathbb{Q}_c^{T_{j-1}}$, using equation (A.0.3) is given by

$$\begin{aligned} \mathbf{Futures}(t; \mathbf{T}) &= P_c(t; T_{j-1}) \mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left[\mathbf{Futures}(T_{j-1}; \mathbf{T}) \right] \\ &= NP_c(t; T_{j-1}) \mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left[1 - L_y(T_{j-1}, T_j) \right] \\ &= NP_c(t; T_{j-1}) \left\{ 1 - \mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left[L_y(T_{j-1}, T_j) \right] \right\}. \end{aligned} \quad (\text{E.2.1})$$

Passing from $\mathbb{Q}_c^{T_{j-1}}$ to $\mathbb{Q}_c^{T_j}$, as in Section E.1, the expectation in equation (E.2.1) becomes

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}_c^{T_{j-1}}} \left[L_y(T_{j-1}, T_j) \right] &= \frac{P_c(t, T_j)}{P_c(t, T_{j-1})} \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\frac{1}{P_c(T_{j-1}, T_j)} L_y(T_{j-1}, T_j) \right] \\ &= \frac{1}{1+F_{c,j}(t)\delta_c(T_{j-1}, T_j)} \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\left(1 + L_c(T_{j-1}, T_j) \delta_c(T_{j-1}, T_j) \right) L_y(T_{j-1}, T_j) \right].\end{aligned}$$

Therefore, we have

$$\mathbf{Futures}(t; \mathbf{T}) = NP_c(t; T_{j-1}) \left\{ 1 - \frac{\mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\left(1 + L_c(T_{j-1}, T_j) \delta_c(T_{j-1}, T_j) \right) L_y(T_{j-1}, T_j) \right]}{1+F_{c,j}(t)\delta_c(T_{j-1}, T_j)} \right\}. \quad (\text{E.2.2})$$

We can see that the price of the Futures depend on the expectation of the product between the two rates $L_c(T_{j-1}, T_j)$ and $L_x(T_{j-1}, T_j)$ under the forward measure $\mathbb{Q}_c^{T_j}$. The expectation depend on the particular model chosen for the joint distribution of these two rates. We have, in general

$$\begin{aligned}R_x^{Fut}(t; \mathbf{T}) &:= \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[L_x(T_{j-1}, T_j) \right] + C_x^{Fut}(t; T_{j-1}) \\ &= F_{x,j}(t) + C_x^{Fut}(t; T_{j-1}).\end{aligned} \quad (\text{E.2.3})$$

E.3 IRS

The payoff of IRSlets are given in equation (2.3.1). Since IRS are OTC products, using formula in equation (A.0.3), we have

$$\begin{aligned}\mathbf{IRSlet}_{\text{float}}(t; T_{j-1}, T_j, L_y) &= P_c(t; T_j) \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\mathbf{IRSlet}_{\text{float}}(T_{j-1}; T_{j-1}, T_j, L_y) \right] \\ &= NP_c(t; T_j) F_{y,j}(t) \delta_L(T_{j-1}, T_j)\end{aligned} \quad (\text{E.3.1})$$

and

$$\begin{aligned}\mathbf{IRSlet}_{\text{fix}}(t; S_{i-1}, S_i, K) &= P_c(t; S_i) \mathbb{E}_t^{\mathbb{Q}_c^{S_i}} \left[\mathbf{IRSlet}_{\text{fix}}(S_{i-1}; S_{i-1}, S_i, K) \right] \\ &= NP_c(t; S_i) K \delta_K(S_{i-1}, S_i).\end{aligned} \quad (\text{E.3.2})$$

Therefore, we have the present values given by

$$\begin{aligned}\mathbf{IRS}_{\text{float}}(t; \mathbf{T}, L_y) &:= \sum_{j=1}^m \mathbf{IRSlet}_{\text{float}}(t; T_{j-1}, T_j, L_y) \\ &= N \sum_{j=1}^m P_c(t; T_j) F_{y,j}(t) \delta_L(T_{j-1}, T_j)\end{aligned} \quad (\text{E.3.3})$$

and

$$\begin{aligned}
 \mathbf{IRS}_{\text{fix}}(t; \mathbf{S}, K) &:= \sum_{i=1}^n \mathbf{IRSlet}_{\text{fix}}(t; S_{i-1}, S_i, K) \\
 &= N \sum_{i=1}^n P_c(t; S_i) K \delta_K(S_{i-1}, S_i) \\
 &= NK A_c(t; \mathbf{S})
 \end{aligned} \tag{E.3.4}$$

where

$$A_c(t; \mathbf{S}) := \sum_{i=1}^n P_c(t; S_i) \delta_K(S_{i-1}, S_i). \tag{E.3.5}$$

Hence, the price of the complete IRS is given by

$$\begin{aligned}
 \mathbf{IRS}(t; \mathbf{T}, \mathbf{S}, L_y, K, \omega) &:= \omega \left[\mathbf{IRSlet}_{\text{float}}(t; \mathbf{T}, L_y) - \mathbf{IRSlet}_{\text{fix}}(t; \mathbf{S}, K) \right] \\
 &= N\omega \left[R_y^{\text{IRS}}(t; \mathbf{T}, \mathbf{S}) - K \right] A_c(t; \mathbf{S}).
 \end{aligned} \tag{E.3.6}$$

The equilibrium IRS rate is the value of K that causes the contract to have zero value at time t , i.e. such that equation (E.3.6) is zero, we have

$$K_{eq} = R_y^{\text{IRS}}(t; \mathbf{T}, \mathbf{S}) := \frac{\sum_{j=1}^m P_c(t; T_j) F_{y,j}(t) \delta_L(T_{j-1}, T_j)}{A_c(t; \mathbf{S})}. \tag{E.3.7}$$

E.4 OIS

The payoff of OIS are given in equation (2.3.6). Using formula in equation (A.0.3), we have

$$\begin{aligned}
 \mathbf{OISlet}_{\text{float}}(t; T_{j-1}, T_j, R_{on}) &= P_c(t; T_j) \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[\mathbf{OISlet}_{\text{float}}(T_{j-1}; T_{j-1}, T_j, R_{on}) \right] \\
 &= NP_c(t; T_j) R_{on}(t; \mathbf{T}_j) \delta_{on}(T_{j-1}, T_j)
 \end{aligned} \tag{E.4.1}$$

and

$$\mathbf{OISlet}_{\text{fix}}(t; T_{j-1}, T_j, K) = \mathbf{IRSlet}_{\text{fix}}(t; T_{j-1}, T_j, K), \tag{E.4.2}$$

where the floating coupon rate is given by

$$\begin{aligned}
 R_{on}(t; \mathbf{T}_j) &:= \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[R_{on}(T_j; \mathbf{T}_j) \right] \\
 &= \frac{1}{\delta_{on}(T_{j-1}, T_j)} \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left\{ \prod_{k=1}^{n_i} \left[1 + R_{on}(T_{i,k-1}, T_{i,k}) \delta_{on}(T_{i,k-1}, T_{i,k}) \right] - 1 \right\} \\
 &= \frac{1}{\delta_{on}(T_{j-1}, T_j)} \left\{ \prod_{k=1}^{n_i} \left[1 + \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[R_{on}(T_{i,k-1}, T_{i,k}) \right] \delta_{on}(T_{i,k-1}, T_{i,k}) \right] - 1 \right\} \\
 &= \frac{1}{\delta_{on}(T_{j-1}, T_j)} \left\{ \prod_{k=1}^{n_i} \left[1 + R_{on}(t; T_{i,k-1}, T_{i,k}) \delta_{on}(T_{i,k-1}, T_{i,k}) \right] - 1 \right\}
 \end{aligned} \tag{E.4.3}$$

where the tower property¹ for nested conditioned expectations have been used and where

$$R_{on}(t; T_{i,k-1}, T_{i,k}) := \mathbb{E}_t^{\mathbb{Q}_c^{T_j}} \left[R_{on}(T_{i,k-1}, T_{i,k}) \right]. \quad (\text{E.4.4})$$

Assuming perfect collateralization, the over night FRA rates $R_{on}(t; T_{i,k-1}, T_{i,k})$ may be considered martingales under the measure $\mathbb{Q}_c^{T_j}$, hence we have (like in the classical single-curve expression (Chapter 2)),

$$R_{on}(t; T_{i,k-1}, T_{i,k}) = \frac{1}{\delta_{on}(T_{i,k-1}, T_{i,k})} \left[\frac{P_c(t; T_{i,k-1})}{P_c(t; T_{i,k})} - 1 \right]. \quad (\text{E.4.5})$$

In this case the over night FRA rates in equation (E.4.1) assumes a simpler expression

$$\begin{aligned} R_{on}(t; \mathbf{T}_j) &= \frac{1}{\delta_{on}(T_{j-1}, T_j)} \left\{ \prod_{k=1}^{n_j} \left[\frac{P_c(t; T_{j,k-1})}{P_c(t; T_{j,k})} \right] - 1 \right\} \\ &= \frac{1}{\delta_{on}(T_{j-1}, T_j)} \left[\frac{P_c(t; T_{j,0})}{P_c(t; T_{j,1})} \frac{P_c(t; T_{j,1})}{P_c(t; T_{j,2})} \dots \frac{P_c(t; T_{j,n_j-1})}{P_c(t; T_{j,n_j})} - 1 \right] \\ &= \frac{1}{\delta_{on}(T_{j-1}, T_j)} \left[\frac{P_c(t; T_{j,0})}{P_c(t; T_{j,n_j})} - 1 \right] \\ &= \frac{1}{\delta_{on}(T_{j-1}, T_j)} \left[\frac{P_c(t; T_{j-1})}{P_c(t; T_j)} - 1 \right] \\ &= F_c(t; T_{j-1}, T_j). \end{aligned} \quad (\text{E.4.6})$$

The OIS legs values are given by

$$\begin{aligned} \text{OIS}_{\text{float}}(t; \mathbf{T}, R_{on}) &:= \sum_{j=1}^m \text{OISlet}_{\text{float}}(t; T_{j-1}, T_j, R_{on}) \\ &= N \sum_{j=1}^m P_c(t; T_j) R_{on}(t; \mathbf{T}_j) \delta_{on}(T_{j-1}, T_j) \\ &= N \sum_{j=1}^m [P_c(t; T_{j-1}) - P_c(t; T_j)] \\ &= N [P_c(t; T_0) - P_c(t; T_m)] \end{aligned} \quad (\text{E.4.7})$$

and

$$\begin{aligned} \text{OIS}_{\text{fix}}(t; \mathbf{S}, K) &:= \sum_{i=1}^n \text{OISlet}_{\text{fix}}(t; S_{i-1}, S_i, K) \\ &= \text{IRS}_{\text{fix}}(t; \mathbf{S}, K) \\ &= N K A_c(t; \mathbf{S}). \end{aligned} \quad (\text{E.4.8})$$

¹If Z is a function of Y then $\mathbb{E}[\mathbb{E}(X|Y)|Z] = \mathbb{E}(X|Z)$.

The price of the complete OIS is thus given by

$$\begin{aligned}\mathbf{OIS}(t; \mathbf{T}, \mathbf{S}, R_{on}^c, K, \omega) &= \omega \left[\mathbf{OIS}_{\text{float}}(t; \mathbf{T}, R_{on}) - \mathbf{OIS}_{\text{fix}}(t; \mathbf{S}, K) \right] \\ &= N\omega \left[R_{on}^{\text{OIS}}(t; \mathbf{T}, \mathbf{S}) - K \right] A_c(t; \mathbf{S}).\end{aligned}\quad (\text{E.4.9})$$

The equilibrium OIS rate is the value of K that causes the contract to have zero value at time t , i.e. such that equation (E.4.9) is zero, we have

$$K_{eq} = R_x^{\text{OIS}}(t; \mathbf{T}, \mathbf{S}) := \frac{P_c(t; T_0) - P_c(t; T_m)}{A_c(t; \mathbf{S})}. \quad (\text{E.4.10})$$

E.5 IRBS

E.5.1 IRBS as single IRS

The schedule and payoff of IRBS as single IRS are given in equation (2.3.13), respectively. The IRBS price is given, using equation (E.3.6) and factorising out the spread in the second floating leg, by

$$\begin{aligned}\mathbf{IRBS}(t; \mathbf{T}_x, \mathbf{T}_y, L_x, L_y, \omega, \Delta_{x,y}) &:= \omega \left[\mathbf{IRS}_{x,\text{float}}(t; \mathbf{T}_x, L_x) \right. \\ &\quad \left. - \mathbf{IRS}_{y,\text{float}}(t; \mathbf{T}_y, L_y, \Delta_{x,y}) \right].\end{aligned}\quad (\text{E.5.1})$$

The equilibrium IRBS spread is given by

$$\Delta(t; \mathbf{T}_x, \mathbf{T}_y) = \frac{\mathbf{IRS}_{x,\text{float}}(t; \mathbf{T}_x, L_x) - \mathbf{IRS}_{y,\text{float}}(t; \mathbf{T}_y, L_y)}{NA_c(t; \mathbf{T}_y)}. \quad (\text{E.5.2})$$

Appendix F

Analysis of the SABR Model

We analyse, here, the mathematical proof of formulae used in Chapter 4. As stated in Chapter 4, Hagan, Kumar, Lesniewski and Woodward (2002) used singular perturbation techniques to price European options under the SABR model. We start by introducing the Singular Perturbation theory and then apply the theory to the SABR Model.

F.1 Singular Perturbation theory

Exact solutions of some problems are impossible to find, thereto approximation methods may be used. Here, we consider perturbation theory. In perturbation theory, a singular perturbation problem is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero. However a regular perturbation problem, is a problem that an approximation can be obtained by simply setting the small parameter to zero.

Example F.1. *Below is an example of algebraic equations and differential equations*

1. *Singular perturbation*

$$\varepsilon x^4 + 2x^3 + 25x^2 + 2014x + 1989 = 0$$

$$\varepsilon y'' + (1 + 2x)y' - 2y = 0 \quad \text{for } 0 < x < 1 \quad y(0) = \varepsilon, \text{ and } y(1) = 1$$

2. *Regular perturbation*

$$x^5 - 4x^3 + \varepsilon x + 3 = 0$$

$$y'' - y' + \varepsilon y^3 = 0 \quad \text{where } y(0) = 0 \text{ and } y(1) = \varepsilon.$$

As it can be inferred from equations above, in singular perturbation, taking $\varepsilon = 0$, changes the very nature of the problem: reduced the possible number of solutions and boundary conditions cannot be satisfied.

Singular perturbation problems occur in a broad array of contexts and the more basic methods used to tackle problems in this field are: matched asymptotic expansions and WKB approximation for spatial problems, and in time, the Poincaré-Lindstedt method, the method of multiple scales and periodic averaging.

We consider one example of singular perturbation problem and apply the method of matched asymptotic expansions to it.

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^x, \quad \text{for } 0 < x < 1 \quad (\text{F.1.1})$$

with boundary conditions $y(0) = 2$, and $y(1) = 1$.

We proceed in 4 steps to construct the first-term approximation of the solution for a small ε .

Step 1: Outer solution

We assume that the solution of equation (F.1.1) can be expanded in powers of ε

$$y \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \cdots \quad (\text{F.1.2})$$

We substitute (F.1.2) into equation (F.1.1), we get

$$\varepsilon(y_0'' + \varepsilon y_1'' + \cdots) + \varepsilon x(y_0' + \varepsilon y_1' + \cdots) - (y_0 + \varepsilon y_1 + \cdots) = -e^x. \quad (\text{F.1.3})$$

$\mathcal{O}(1)$ equation, all terms without ε , gives: $-y_0 = -e^x$. It follows that

$$y_0 = e^x. \quad (\text{F.1.4})$$

This function cannot satisfy the boundary condition $y(0) = 2$, and $y(1) = 1$, it suggests that the solution in equation (F.1.4) and equation (F.1.2) are not able to describe the solution over $0 \leq x \leq 1$. We assume: firstly that equation (F.1.4) describes the solution over most of the interval, but there is a boundary layer at either $x = 0$ or $x = 1$, where a different approximation must be used. Secondly, we also assume that we have a boundary layer at $x = 0$. We do it in the next step, solution (F.1.4) is the first term in the expansion of the **outer solution**.

Step 2: Boundary layers and Matching

Because of the boundary layer at $x = x_0 = 0$, we rescale the variable x by introducing a local variable ξ given by

$$\xi = \frac{x - x_0}{\delta(\varepsilon)} = \frac{x}{\varepsilon^\lambda}. \quad (\text{F.1.5})$$

We consider ξ as a function of x only, this has the effect of stretching the region near $x = 0$ as ε becomes small; that is why, equation (F.1.5) is sometimes referred to as a stretching transformation. From the chain rule, for a given function $f(\xi)$, we have

$$\frac{df(\xi)}{dx} = \frac{d\xi}{dx} \frac{df(\xi)}{d\xi} = \frac{1}{\varepsilon^\lambda} \frac{df(\xi)}{d\xi} \quad \text{it implies} \quad \frac{d^2f(\xi)}{dx^2} \rightarrow \frac{1}{\varepsilon^{2\lambda}} \frac{d^2f(\xi)}{d\xi^2}.$$

It follows that

$$\varepsilon^{2-2\lambda} \frac{d^2Y}{d\xi^2} + \varepsilon\xi \frac{dY}{d\xi} - Y = -e^{\varepsilon^\lambda \xi}. \quad (\text{F.1.6})$$

We know that

$$e^{\varepsilon^\lambda \xi} \sim 1 + \varepsilon^\lambda \xi + \dots$$

The appropriate expansion for the boundary-layer solution is now

$$Y \sim Y_0(\xi) + \varepsilon^\theta Y_1(\xi) + \dots \quad (\text{F.1.7})$$

where $\theta > 0$. We determine the correct balancing in equation (F.1.6) which contains three terms. In step 1, we have considered the balance between the third and fourth term, so it remains others possibilities:

- term 1 \sim term 3 and term 2 is of higher order in ε
term 1 \sim term 3 requires that $2 - 2\lambda = 0 \implies \lambda = 1$. Now term 3 is $\mathcal{O}(0)$ and term 2 is $\mathcal{O}(\varepsilon)$, that matches the assumption of term 2 is of higher order in ε . we have the balance needed, we ignore the rest of possibilities.

Substituting $\lambda = 1$, and equation (F.1.7) in equation (F.1.6) and the boundary condition at $\xi = 0$. Considering $\mathcal{O}(1)$, we have that

$$\begin{aligned} Y_0'' - Y_0' &= -1 \quad \text{for } 0 < \xi < \infty \\ Y_0(0) &= 2. \end{aligned}$$

The general solution of this problem is

$$Y_0(\xi) = 1 + Ke^{-\xi} + (1 - K)e^\xi \quad (\text{F.1.8})$$

where K is an arbitrary constant. To determine K , we note that, both the inner and outer expansions are approximations of the same function. Therefore, in the region between the inner and outer layers we should expect that the two expansions will give the same result. We require that

$$\lim_{\xi \rightarrow \infty} Y_0 = \lim_{x \rightarrow 0} y_0 \quad (\text{F.1.9})$$

From this, we get $K = 1$, hence $Y_0(\xi) = 1 + Ke^{-\xi}$. The solution, $Z(\eta)$, in the boundary layer at $x = 1$, is determined by introducing the local variable

$$\eta = \frac{x-1}{\varepsilon^\theta}. \quad (\text{F.1.10})$$

Using (F.1.10) and following the steps above about chain rules and introducing into (F.1.1), we get

$$\varepsilon^{2-2\theta} \frac{d^2 Z}{d\eta^2} + (1 + \varepsilon\eta)\varepsilon^{1-\theta} \frac{dZ}{d\eta} - Z = -e^{1+\varepsilon^\theta\eta}. \quad (\text{F.1.11})$$

The balance occurs for $\theta = 1$, as the steps above, the expansion $Z \sim Z_0$ yields the problem

$$\begin{aligned} Z_0'' + Z_0' - Z_0 &= -e \quad \text{for } -\infty < \xi < 0 \\ Z_0(0) &= 1. \end{aligned}$$

The general solution of this problem is

$$Z_0(\eta) = e + De^{\frac{-1+\sqrt{5}}{2}\eta} + (1 - e - D)e^{\frac{-1-\sqrt{5}}{2}\eta} \quad (\text{F.1.12})$$

where D is an arbitrary constant. To determine D , we require that $Z_0(-\infty) = y_0(1)$. From (F.1.12), we get $D = 1 - e$.

Step 4: Composite Expansion

The last step is to combine them into a single expression, we just add the approximations together and then subtract the part that is common to both. The result is

$$\begin{aligned} y &\sim y_0(x) + Y_0(\xi) - Y_0(\infty) + Z_0(\eta) - Z_0(-\infty) \\ &\sim e^x + e^{-x/\varepsilon} + (1 - e)e^{(x-1)(-1+\sqrt{5})/(2\varepsilon)}. \end{aligned}$$

This example and more details can be find in (Holmes, 2012).

F.2 Scaling

Hagan considered small volatility $\hat{\alpha}$ and volvol ν , thence re-write $\hat{\alpha} \rightarrow \varepsilon\hat{\alpha}$ and $\hat{\nu} \rightarrow \varepsilon\hat{\nu}$ to the problem:

$$dF = \hat{\alpha}C(F)dW_1, \quad F(0) = f \quad (\text{F.2.1})$$

$$d\hat{\alpha} = \hat{\nu}\hat{\alpha}dW_2, \quad \hat{\alpha}(0) = \alpha \quad (\text{F.2.2})$$

with

$$dW_1dW_2 = \rho dt \quad (\text{F.2.3})$$

in the limit $\varepsilon \ll 1$.

F.3 Application of perturbation theory to SABR model

The scaling in the previous section gave the possibility to use perturbation theory to the SABR model, and the analyse was carried out by using a general $C(\hat{F})$ and the results were applied to $C(\hat{F}) = \hat{F}^\beta$. We scale as follows:

$$\sigma = \frac{\hat{\alpha}}{\varepsilon} \iff \hat{\alpha} = \varepsilon\sigma \quad \text{and} \quad \nu = \frac{\hat{\nu}}{\varepsilon} \iff \hat{\nu} = \varepsilon\nu. \quad (\text{F.3.1})$$

This leads to:

$$dF = \varepsilon\sigma C(F)dW_1, \quad F(0) = f \quad (\text{F.3.2})$$

$$d\sigma = \varepsilon\nu\sigma dW_2, \quad \sigma(0) = \varepsilon\alpha. \quad (\text{F.3.3})$$

We first use the forward Kolmogorov equation to simplify the option pricing problem. Suppose the economy is in the state $\hat{F}(t) = f, \hat{\alpha}(t) = \alpha$ at date t . Define the probability density $p(t, f, \alpha; T, F, A)$ by

$$p(t, f, \alpha, T, F, A)dFdA = \text{prob}\{F < \hat{F}(T) < F + dF, A < \sigma(\hat{T}) < A + dA | \hat{F}(t) = f, \hat{\sigma}(t) = \alpha\}. \quad (\text{F.3.4})$$

We consider the following propositions¹.

Proposition F.2 (Kolmogorov forward equation). *Let consider the stochastic differential equation*

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u). \quad (\text{F.3.5})$$

For $0 \leq t < T$, let $p(t, T, x, y)$ be a transition density for the solution of this equation². Assume that $p(t, T, x, y) = 0$ for $0 \leq t \leq T$ and $y \leq 0$. We show that $p(t, T, x, y)$ satisfies the Kolmogorov forward equation

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}\left(\beta(t, y)p(t, T, x, y)\right) + \frac{1}{2}\frac{\partial}{\partial y^2}\left(\gamma^2(t, y)p(t, T, x, y)\right). \quad (\text{F.3.6})$$

Below is the proof in 5 steps.

Proof. 1. Let $b > 0$ and h_b be a function with continuous first and second derivatives such that $h_b(x) = 0$ for all $x \leq 0$, $h'_b(x) = 0$ for all $x \geq b$, and $h_b(b) = h'_b(b) = 0$. Let $X(u)$ be a solution of equation (F.3.5) with initial condition $X(t) = x \in (0, b)$. We use Itô's formula to compute $dh_b(X(u))$.

¹These are exercises in (Shreve, 2004)

²if we solve equation (F.3.5) with initial condition $X(t) = x$, then the random variable $X(T)$ has density function $p(t, T, x, y)$ in y variable.

2. Let $0 \leq t < T$ be given. We integrate the equation we obtain in step (1) from t to T . We take expectations and we use the fact that $X(u)$ has density function $p(t, T, x, y)$ in the y -variable, we get

$$\begin{aligned} \int_0^b h_b(y)p(t, T, x, y)dy &= h_b(x) + \int_t^T \int_0^b \beta(u, y)p(t, u, x, y)h'_b(y)dydu \\ &+ \frac{1}{2} \int_t^T \int_0^b \gamma^2(u, y)p(t, u, x, y)h''_b(y)dydu. \end{aligned} \quad (\text{F.3.7})$$

3. We integrate the integrals $\int_0^b \cdots dy$ on the right hand side of equation (F.3.7) by parts and we use the fact that $h_b(b) = h'_b(b) = 0$ and $h_b(0) = h'_b(0) = 0$, we obtain

$$\begin{aligned} \int_0^b h_b(y)p(t, T, x, y)dy &= h_b(x) - \int_t^T \int_0^b \frac{\partial}{\partial y} [\beta(u, y)p(t, u, x, y)] h_b(y)dydu \\ &+ \frac{1}{2} \int_t^T \int_0^b \frac{\partial^2}{\partial y^2} [\gamma^2(u, y)p(t, u, x, y)] h_b(y)dydu. \end{aligned} \quad (\text{F.3.8})$$

4. We differentiate the equation (F.3.8) with respect to T , we get

$$\begin{aligned} \int_0^b h_b(y) \left(\frac{\partial}{\partial T} p(t, T, x, y) + \frac{\partial}{\partial y} [\beta(T, y)p(t, T, x, y)] \right. \\ \left. - \frac{1}{2} \frac{\partial^2}{\partial y^2} [\gamma^2(T, y)p(t, T, x, y)] \right) dy = 0. \end{aligned} \quad (\text{F.3.9})$$

5. Using equation (F.3.9), we show that there cannot be numbers $0 < y_1 < y_2$ such that

$$\begin{aligned} \frac{\partial}{\partial T} p(t, T, x, y) + \frac{\partial}{\partial y} [\beta(T, y)p(t, T, x, y)] \\ - \frac{1}{2} \frac{\partial^2}{\partial y^2} [\gamma^2(T, y)p(t, T, x, y)] > 0 \quad \text{for all } y \in (y_1, y_2). \end{aligned}$$

Likewise, there cannot be numbers $0 < y_1 < y_2$ such that

$$\begin{aligned} \frac{\partial}{\partial T} p(t, T, x, y) + \frac{\partial}{\partial y} [\beta(T, y)p(t, T, x, y)] \\ - \frac{1}{2} \frac{\partial^2}{\partial y^2} [\gamma^2(T, y)p(t, T, x, y)] < 0 \quad \text{for all } y \in [y_1, y_2]. \end{aligned}$$

Obviously, if

$$\frac{\partial}{\partial T} p(t, T, x, y) + \frac{\partial}{\partial y} [\beta(T, y)p(t, T, x, y)] - \frac{1}{2} \frac{\partial^2}{\partial y^2} [\gamma^2(T, y)p(t, T, x, y)]$$

is a continuous function of y , then this expression must be zero for every $y > 0$, and hence $p(t, T, x, y)$ satisfies the Kolmogorov forward equation stated in the proposition (equation (F.3.6)).

□

As a function of forward variables T , F , A , using Proposition F.2, the density p satisfies the following forward Kolmogorov equation

$$\begin{aligned} \frac{\partial}{\partial T} p(t, f, \alpha, T, F, A) &= \frac{1}{2} \frac{\partial^2}{\partial F^2} [\varepsilon^2 A^2 C^2(F) p] + \frac{1}{2} \frac{\partial^2}{\partial A^2} [\varepsilon^2 \nu^2 A^2 p] \\ &\quad + \frac{1}{2} \frac{\partial^2}{\partial F \partial A} [\rho \varepsilon^2 \nu A^2 C(F) p] + \frac{1}{2} \frac{\partial^2}{\partial A \partial F} [\rho \varepsilon^2 \nu A^2 C(F) p]. \end{aligned}$$

We get

$$\begin{aligned} \frac{\partial}{\partial T} p(t, f, \alpha, T, F, A) &= \frac{1}{2} \varepsilon^2 A^2 \frac{\partial^2}{\partial F^2} [C^2(F) p] + \rho \varepsilon^2 \nu \frac{\partial^2}{\partial F \partial A} [A^2 C(F) p] \\ &\quad + \frac{1}{2} \varepsilon^2 \nu^2 \frac{\partial^2}{\partial A^2} [A^2 p] \quad \text{for } t < T, \end{aligned} \quad (\text{F.3.10})$$

with

$$p = \delta(F - f) \delta(A - \alpha) \quad \text{at } T = t. \quad (\text{F.3.11})$$

Let $V(t, f, \alpha)$ be the value of a European call option at date t , when the economy is in state $F(t) = f$, $\sigma(t) = \alpha$. Let t_{ex} be the option's exercise date, and let K be its strike. Omitting the discount factor $P(t, t_{ex})$, because we do a current valuation of the final payoff, the value of the option is

$$\begin{aligned} V(t, f, \alpha) &= \mathbb{E} \left[\left(F(t_{ex}) - K \right)^+ \middle| F(t) = f, \sigma(t) = \alpha \right] \\ &= \int_{A=-\infty}^{\infty} \int_{F=K}^{\infty} (F - K) p(t, f, \alpha; t_{ex}, F, A) dF dA. \end{aligned} \quad (\text{F.3.12})$$

Since

$$p(t, f, \alpha, t_{ex}, F, A) = \delta(F - f) \delta(A - \alpha) + \int_t^{t_{ex}} \frac{\partial}{\partial T} p(t, f, \alpha, T, F, A) dT.$$

Equation (F.3.12) can be rewritten as

$$V(t, f, \alpha) = (f - K)^+ + \int_t^{t_{ex}} \int_K^{\infty} \int_{-\infty}^{\infty} (F - K) \frac{\partial}{\partial T} p(t, f, \alpha, T, F, A) dA dF dT. \quad (\text{F.3.13})$$

We substitute (F.3.10) for $\frac{\partial}{\partial T} p$ into (F.3.13). Integrating the A derivatives $\rho \varepsilon^2 \nu \frac{\partial^2}{\partial F \partial A} [A^2 C(F) p]$ and $\frac{1}{2} \varepsilon^2 \nu^2 \frac{\partial^2}{\partial A^2} [A^2 p]$ over all A yields zero. Therefore our option price reduces to

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \varepsilon^2 \int_t^{t_{ex}} \int_{-\infty}^{\infty} \int_K^{\infty} (F - K) A^2 \frac{\partial^2}{\partial F^2} [C^2(F) p] dF dA dT$$

where, the order of integration has been changed. Integrating by parts twice with respect to F yields

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \varepsilon^2 C^2(K) \int_{T=t}^{t_{ex}} \int_{A=-\infty}^{\infty} A^2 p(t, f, \alpha, T, K, A) dA dT.$$

We further simplify the problem by defining

$$P(t, f, \alpha, T, K) := \int_{-\infty}^{\infty} A^2 p(t, f, \alpha, T, K, A) dA. \quad (\text{F.3.14})$$

Before proceeding, we need to consider the following proposition.

Proposition F.3 (Kolmogorov backward equation). *Let consider the stochastic differential equation*

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u). \quad (\text{F.3.15})$$

For $0 \leq t < T$, let $P(t, T, x, y)$ be a transition density for the solution of this equation³. Assuming that $P(t, T, x, y) = 0$ for $0 \leq t \leq T$ and $y \leq 0$, we show that $P(t, T, x, y)$ satisfies the Kolmogorov Backward equation

$$-\frac{\partial}{\partial t}P(t, T, x, y) = \beta(t, x)\frac{\partial}{\partial x}P(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)\frac{\partial^2}{\partial x^2}P(t, T, x, y). \quad (\text{F.3.16})$$

The proof of this proposition depends on the following theorem.

Theorem F.4 (Feynman-Kac). *Let consider the stochastic differential equation*

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u). \quad (\text{F.3.17})$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$ and let $t \in [0, T]$ be given. Define the function

$$g(t, x) = \mathbb{E}^{t,x}h(X(T)). \quad (\text{F.3.18})$$

(we assume that $\mathbb{E}^{t,x}|h(X(T))| < \infty$ for all t and x .) Then $g(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t}g(t, x) + \beta(t, x)\frac{\partial}{\partial x}g(t, x) + \frac{1}{2}\gamma^2(t, x)\frac{\partial^2}{\partial x^2}g(t, x) = 0 \quad (\text{F.3.19})$$

and the terminal condition

$$g(T, x) = h(x) \quad \text{for all } x. \quad (\text{F.3.20})$$

Proof. To prove the Feynman-Kac Theorem F.4, we assume that $g(t, X(t))$ is a martingale⁴. Let $X(t)$ be a solution of (F.3.17). Using Itô's formula, we compute $dg(t, X(t))$ as follows (we omit the argument (t, x))

$$\begin{aligned} dg(t, X(t)) &= \frac{\partial g}{\partial t}dt + \beta \frac{\partial g}{\partial x}dt + \gamma \frac{\partial g}{\partial x}dW + \frac{1}{2}\gamma^2 \frac{\partial^2 g}{\partial x^2}dt \\ &= \left[\frac{\partial}{\partial t}g + \beta \frac{\partial}{\partial x}g + \frac{1}{2}\gamma^2 \frac{\partial^2}{\partial x^2}g \right]dt + \gamma \frac{\partial g}{\partial x}dW. \end{aligned}$$

³if we solve (F.3.15) with initial condition $X(t) = x$, then the random variable $X(T)$ has density function $P(t, T, x, y)$ in y variable.

⁴ $g(t, X(t))$ is a martingale has been proved in (Shreve, 2004)

Since $g(t, X(t))$ is a martingale, it follows that the net dt term in $dg(t, X(t))$ is zero, (we put back the argument (t, x))

$$\frac{\partial}{\partial t}g(t, x) + \beta(t, x)\frac{\partial}{\partial x}g(t, x) + \frac{1}{2}\gamma^2(t, x)\frac{\partial^2}{\partial x^2}g(t, x) = 0.$$

Hence the theorem. \square

With Theorem F.4, we can prove Proposition F.3.

proof of Proposition F.3. Let $g(t, x)$ be defined by

$$g(t, x) = \mathbb{E}^{t, xh(X(T))} = \int_0^\infty h(y)p(t, T, x, y)dy \quad (\text{F.3.21})$$

for any function $h(y)$. From Theorem F.4, the function $g(t, x)$ satisfies the partial differential equation

$$\frac{\partial}{\partial t}g(t, x) + \beta(t, x)\frac{\partial}{\partial x}g(t, x) + \frac{1}{2}\gamma^2(t, x)\frac{\partial^2}{\partial x^2}g(t, x) = 0. \quad (\text{F.3.22})$$

Substituting (F.3.21) in (F.3.22), we get

$$\int_0^\infty h(y) \left[\frac{\partial}{\partial t}p(t, T, x, y) + \beta(t, x)\frac{\partial}{\partial x}p(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)\frac{\partial^2}{\partial x^2}p(t, T, x, y) \right] = 0. \quad (\text{F.3.23})$$

If $h = 0$, it implies that $g = 0$, and this does not make sense, so we reject the case $h = 0$, the only way (F.3.23) can hold for any h is for P satisfying (F.3.17). \square

From (F.3.15), using Proposition F.3, then P satisfies the Backward's Kolmogorov Equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\varepsilon^2\alpha^2C^2(f)\frac{\partial^2 P}{\partial f^2} + \rho\varepsilon^2\nu\alpha^2C(f)\frac{\partial^2 P}{\partial f\partial\alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2\frac{\partial^2 P}{\partial\alpha^2} = 0 \quad \text{for } t < T$$

and

$$P = \alpha^2\delta(f - K) \quad \text{for } t = T.$$

Since P does not appear explicitly in this equation, P depends only on the combination $T - t$, and not on t and T separately. So we define

$$\tau := T - t \quad \text{and} \quad \tau_{ex} := t_{ex} - t.$$

Then our pricing equation becomes

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2}\varepsilon^2C^2(K) \int_0^{\tau_{ex}} P(\tau, f, \alpha, K)d\tau \quad (\text{F.3.24})$$

where $P(\tau, f, \alpha; K)$ is the solution of the problem

$$\begin{aligned} \frac{\partial P}{\partial \tau} &= \frac{1}{2}\varepsilon^2\alpha^2C^2(f)\frac{\partial^2 P}{\partial f^2} + \rho\varepsilon^2\nu\alpha^2C(f)\frac{\partial^2 P}{\partial f\partial\alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2\frac{\partial^2 P}{\partial\alpha^2} \quad \text{for } \tau > 0 \\ P &= \alpha^2\delta(f - K), \quad \text{for } \tau = 0. \end{aligned} \quad (\text{F.3.25})$$

We solve (F.3.25) to obtain $P(\tau, f, \alpha; K)$ and then substitute this solution into (F.3.24) to get the value of $V(t, f, \alpha)$. This yields the price under the SABR model. In order to solve (F.3.25), we need to transform it into a heat equation and use the result in equation (G.0.7). Let us make the following change of variable

$$\xi = \frac{f - K}{\varepsilon}.$$

Considering the Taylor expansion of C on ξ , we get

$$C(f) = C(K + \varepsilon\xi) = C_0\{1 + \varepsilon\gamma_1\xi + \frac{1}{2}\varepsilon^2\gamma_2\xi^2 + \dots\}$$

where

$$C_0 = C(K), \quad \gamma_1 = \frac{C'(K)}{C(K)}, \quad \gamma_2 = \frac{C''(K)}{C(K)}.$$

Equation (F.3.25) becomes

$$\begin{aligned} \frac{\partial P}{\partial \tau} &= \frac{1}{2}\alpha^2C_0^2\{1 + 2\varepsilon\gamma_1\xi + \varepsilon^2(\gamma_2 + \gamma_1^2)\xi^2 + \dots\}\frac{\partial^2 P}{\partial\xi^2} + \\ &\rho\varepsilon\nu\alpha^2\{1 + \varepsilon\gamma_1\xi + \frac{1}{2}\varepsilon^2\gamma_2\xi^2 + \dots\}\frac{\partial^2 P}{\partial\xi\partial\alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2\frac{\partial^2 P}{\partial\alpha^2} \quad \text{for } \tau > 0 \quad (\text{F.3.26}) \\ P &= \alpha^2\delta(\varepsilon\xi) = \frac{\alpha^2}{\varepsilon}\delta(\xi) \quad \text{as } \tau \rightarrow 0. \end{aligned}$$

Expanding P as

$$P_\varepsilon(t, \xi, \alpha) = \frac{1}{\varepsilon}P_0(t, \xi, \alpha) + P_1(t, \xi, \alpha) + \varepsilon P_2(t, \xi, \alpha) + \varepsilon^2 P_3(t, \xi, \alpha) + \dots$$

Substituting this expansion into (F.3.26) and equating like powers of ε , leads to the following $\mathcal{O}(\frac{1}{\varepsilon})$ problem:

$$\begin{aligned} \frac{\partial P_0}{\partial \tau} &= \frac{1}{2}\alpha^2C_0^2\frac{\partial^2 P_0}{\partial\xi^2} \quad \text{for } \tau > 0 \\ P_0 &= \alpha^2\delta(\xi) \quad \text{as } \tau \rightarrow 0. \end{aligned} \quad (\text{F.3.27})$$

To compute the solution of (F.3.27), we consider the following change of variables

$$y := \frac{\xi}{\alpha^2}, \quad k = \frac{C_0^2}{2\alpha^2}, \quad t := \tau, \quad \text{and} \quad u(t, y) := P_0(\tau, \xi)$$

The problem in equation (F.3.27) is then reduced to

$$\begin{aligned} \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial y^2} &= 0 \\ u(t = 0, y) &= \delta(y). \end{aligned} \tag{F.3.28}$$

Equation (F.3.28) is the heat or diffusion equation, subject to a Dirac delta in the initial condition. Taking $f = 0$ in equation (G.0.7), the solution of equation (F.3.28) is given by

$$u(t, y) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{y^2}{4kt}\right).$$

Turning back to the original variables, we get

$$P_0(\tau, \xi) = \frac{|\alpha|}{\sqrt{2\pi C_0^2 \tau}} \exp\left(-\frac{(f - K)^2}{2\alpha^2 C_0^2 \xi^2 \tau}\right). \tag{F.3.29}$$

Similarly, like powers of ε leads to the following $\mathcal{O}(1)$ problem:

$$\begin{aligned} \frac{\partial P_1}{\partial \tau} - \frac{1}{2}\alpha^2 C_0^2 \frac{\partial^2 P_1}{\partial \xi^2} &= \alpha^2 \gamma_1 C_0^2 \xi \frac{\partial^2 P_0}{\partial \xi^2} \quad \text{for } \tau > 0 \\ P_1 &= 0 \quad \text{as } \tau \rightarrow 0, \end{aligned} \tag{F.3.30}$$

whereas the $\mathcal{O}(\varepsilon)$ problem is given by

$$\begin{aligned} \frac{\partial P_2}{\partial \tau} - \frac{1}{2}\alpha^2 C_0^2 \frac{\partial^2 P_2}{\partial \xi^2} &= \alpha^2 \gamma_1 C_0^2 \xi \frac{\partial^2 P_1}{\partial \xi^2} + \rho \nu C_0 \frac{\partial^2 P_0}{\partial \xi \partial \alpha} \quad \text{for } \tau > 0 \\ P_2 &= 0 \quad \text{as } \tau \rightarrow 0. \end{aligned} \tag{F.3.31}$$

Substituting (F.3.29) into equation (F.3.30) and making some change of variable as we did with $\mathcal{O}(\frac{1}{\varepsilon})$ problem, the $\mathcal{O}(1)$ problem leads to an inhomogeneous heat equation which is then solved by equation (G.0.7). We also solve the $\mathcal{O}(\varepsilon)$ problem in the same way. We do not give the results of these solutions here, because there are not useful. Hence

$$P = \frac{1}{\varepsilon} P_0 + P_1 + \varepsilon P_2 + \varepsilon^2 P_3 + \dots = \frac{\alpha}{\sqrt{2\pi C_0^2 \varepsilon^2 \tau}} \exp\left(-\frac{(f - K)^2}{2\alpha^2 C_0^2 \varepsilon^2 \tau}\right) + \dots$$

or

$$P = \frac{\alpha}{\sqrt{2\pi C^2(K)\varepsilon^2\tau}} \exp\left(-\frac{(f-K)^2}{2\alpha^2 C^2(K)\varepsilon^2\tau}\right) \{1 + \dots\}.$$

We can write the argument of the exponential of P in form of integral

$$\frac{(f-K)^2}{2\alpha^2 C^2(K)\varepsilon^2\tau} \{1 + \dots\} = \frac{1}{2\tau} \left(\frac{1}{\varepsilon\alpha} \int_K^f \frac{dx}{C(x)} \right)^2 \{1 + \dots\}. \quad (\text{F.3.32})$$

Hence equation (F.3.32) results. Let us define a new variable z by

$$z := \frac{1}{\varepsilon\alpha} \int_K^f \frac{dx}{C(x)}.$$

We have

$$z = \frac{1}{\varepsilon} \frac{J(f) - J(K)}{\alpha}$$

where $J'(x) = 1/C(x)$. Moreover, we define B by

$$B(\varepsilon\alpha z) := C(f).$$

We summarise all the results as follows

$$\begin{aligned} \frac{\partial}{\partial f} &\rightarrow \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)} \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial \alpha} \rightarrow \frac{\partial}{\partial \alpha} - \frac{z}{\alpha} \frac{\partial}{\partial z} \\ \frac{\partial^2}{\partial f^2} &\rightarrow \frac{1}{\varepsilon^2\alpha^2 B^2(\varepsilon\alpha z)} \left\{ \frac{\partial^2}{\partial z^2} - \varepsilon\alpha \frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} \frac{\partial}{\partial z} \right\} \\ \frac{\partial^2}{\partial f \partial \alpha} &\rightarrow \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)} \left\{ \frac{\partial^2}{\partial z \partial \alpha} - \frac{z}{\alpha} \frac{\partial^2}{\partial z^2} - \frac{1}{\alpha} \frac{\partial}{\partial z} \right\} \\ \frac{\partial^2}{\partial \alpha^2} &\rightarrow \frac{\partial^2}{\partial \alpha^2} - \frac{2z}{\alpha} \frac{\partial^2}{\partial z \partial \alpha} + \frac{z^2}{\alpha^2} \frac{\partial^2}{\partial z^2} + \frac{2z}{\alpha^2} \frac{\partial}{\partial z}. \end{aligned} \quad (\text{F.3.33})$$

We have $f - K = \varepsilon\alpha z C(K)$, therefore

$$\delta(f - K) = \delta(\varepsilon\alpha z C(K)) = \frac{1}{\varepsilon\alpha C(K)} \delta(z).$$

Consequently, equation (F.3.24) becomes

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \varepsilon^2 C^2(K) \int_0^{\tau_{ex}} P(\tau, z, \alpha) d\tau$$

where $P(\tau, z, \alpha)$ is the solution of the following boundary value problem

$$\begin{aligned} \frac{\partial P}{\partial \tau} &= \frac{1}{2} (1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2) \frac{\partial^2 P}{\partial z^2} - \frac{1}{2} \varepsilon\alpha \frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} \frac{\partial P}{\partial z} \\ &+ (\varepsilon\rho\nu - \varepsilon^2\nu^2 z) \left(\alpha \frac{\partial^2 P}{\partial z \partial \alpha} - \frac{\partial P}{\partial z} \right) + \frac{1}{2} \varepsilon^2 \nu^2 \alpha^2 \frac{\partial^2 P}{\partial \alpha^2}, \quad \text{for } \tau > 0 \\ P &= \frac{\alpha}{\varepsilon C(K)} \delta(z) \quad \text{as } \tau \rightarrow 0. \end{aligned}$$

Let us define $\hat{P}(\tau, z, \alpha)$ by

$$\hat{P}(\tau, z, \alpha) := \frac{\varepsilon}{\alpha} C(K) P.$$

Consequently, we have

$$\begin{aligned} \frac{\varepsilon C(K)}{\alpha} \frac{\partial P}{\partial \alpha} &= \frac{\varepsilon C(K)}{\alpha} \frac{\partial}{\partial \alpha} \left(\frac{\alpha}{\varepsilon C(K)} \hat{P} \right) = \frac{1}{\alpha} \hat{P} + \frac{\partial \hat{P}}{\partial \alpha} \\ \frac{\varepsilon C(K)}{\alpha} \frac{\partial^2 P}{\partial \alpha^2} &= \frac{\varepsilon C(K)}{\alpha} \frac{\partial^2}{\partial \alpha^2} \left(\frac{\alpha}{\varepsilon C(K)} \hat{P} \right) = \frac{1}{\alpha} \frac{\partial}{\partial \alpha} \left(\hat{P} + \alpha \frac{\partial \hat{P}}{\partial \alpha} \right). \end{aligned}$$

It follows that

$$\frac{\varepsilon C(K)}{\alpha} \frac{\partial^2 P}{\partial \alpha^2} = \frac{1}{\alpha} \frac{\partial}{\partial \alpha} \left(\frac{\partial \hat{P}}{\partial \alpha} + \hat{P} + \alpha \frac{\partial^2 \hat{P}}{\partial \alpha^2} \right) = \frac{2}{\alpha} \frac{\partial \hat{P}}{\partial \alpha} + \frac{\partial^2 \hat{P}}{\partial \alpha^2}.$$

In terms of \hat{P} , we get

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \alpha \varepsilon C(K) \int_0^{\tau_{ex}} \hat{P}(\tau, z, \alpha) d\tau \quad (\text{F.3.34})$$

where \hat{P} is solution of

$$\begin{aligned} \frac{\partial \hat{P}}{\partial \tau} &= \frac{1}{2} (1 - 2\varepsilon \rho \nu z + \varepsilon^2 \nu^2 z^2) \frac{\partial^2 \hat{P}}{\partial z^2} - \frac{1}{2} \varepsilon \alpha \frac{B'(\varepsilon \alpha z)}{B(\varepsilon \alpha z)} \frac{\partial \hat{P}}{\partial z} \\ &+ (\varepsilon \rho \nu - \varepsilon^2 \nu^2 z) \alpha \frac{\partial^2 \hat{P}}{\partial z \partial \alpha} + \frac{1}{2} \varepsilon^2 \nu^2 \left(2\alpha \frac{\partial \hat{P}}{\partial \alpha} + \alpha^2 \frac{\partial^2 \hat{P}}{\partial \alpha^2} \right) \quad \text{for } \tau > 0 \\ \hat{P} &= \delta(z) \quad \text{as } \tau \rightarrow 0. \end{aligned} \quad (\text{F.3.35})$$

Taking all the $\mathcal{O}(1)$ terms, we realise that \hat{P} is the solution of the standard diffusion problem

$$\begin{aligned} \frac{\partial \hat{P}}{\partial \tau} &= \frac{1}{2} \frac{\partial^2 \hat{P}}{\partial z^2} \quad \text{for } \tau > 0 \\ \hat{P} &= \delta(z) \quad \text{as } \tau \rightarrow 0. \end{aligned} \quad (\text{F.3.36})$$

This can be solved using equation (G.0.7). So the solution is a function of z and α . Consequently we can write \hat{P} as

$$\hat{P}(\tau, z, \alpha) = \hat{P}_0(\tau, z) + \varepsilon \hat{P}_1(\tau, z, \alpha) + \varepsilon^2 \hat{P}_2(\tau, z, \alpha) + \dots \quad (\text{F.3.37})$$

For this very reason, the derivatives $\frac{\partial \hat{P}}{\partial \alpha}$, $\frac{\partial \hat{P}}{\partial \alpha}$ and $\frac{\partial \hat{P}}{\partial \alpha}$ are all at least $\mathcal{O}(\varepsilon)$. Substituting (F.3.37) into (F.3.35), we get

$$\begin{aligned} \frac{\partial \hat{P}_0}{\partial \tau} + \varepsilon \frac{\partial \hat{P}_1}{\partial \tau} + \varepsilon^2 \frac{\partial \hat{P}_2}{\partial \tau} + \dots &= \frac{1}{2}(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2) \left[\frac{\partial^2 \hat{P}_0}{\partial z^2} + \varepsilon \frac{\partial^2 \hat{P}_1}{\partial z^2} \right. \\ &\quad \left. + \varepsilon^2 \frac{\partial^2 \hat{P}_2}{\partial z^2} + \dots \right] - \frac{1}{2}\varepsilon\alpha \frac{B'}{B} \left[\frac{\partial \hat{P}_0}{\partial z} + \varepsilon \frac{\partial \hat{P}_1}{\partial z} + \varepsilon^2 \frac{\partial \hat{P}_2}{\partial z} + \dots \right] + (\varepsilon\rho\nu - \\ &\quad \varepsilon^2\nu^2 z)\alpha \left[\frac{\partial^2 \hat{P}_0}{\partial z \partial \alpha} + \varepsilon \frac{\partial^2 \hat{P}_1}{\partial z \partial \alpha} + \varepsilon^2 \frac{\partial^2 \hat{P}_2}{\partial z \partial \alpha} + \dots \right] + \frac{1}{2}\varepsilon^2\nu^2 \left(2\alpha \left[\frac{\partial \hat{P}_0}{\partial \alpha} + \varepsilon \frac{\partial \hat{P}_1}{\partial \alpha} \right. \right. \\ &\quad \left. \left. + \varepsilon^2 \frac{\partial \hat{P}_2}{\partial \alpha} + \dots \right] + \alpha^2 \left[\frac{\partial^2 \hat{P}_0}{\partial \alpha^2} + \varepsilon \frac{\partial^2 \hat{P}_1}{\partial \alpha^2} + \varepsilon^2 \frac{\partial^2 \hat{P}_2}{\partial \alpha^2} + \dots \right] \right) \quad \text{for } \tau > 0. \\ \hat{P} &= \delta(z) \quad \text{as } \tau \rightarrow 0. \end{aligned}$$

We are only solving for \hat{P} through $\mathcal{O}(\varepsilon^2)$, therefore, we have

$$\begin{aligned} \frac{\partial \hat{P}}{\partial \tau} &= \frac{1}{2}(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2) \frac{\partial^2 \hat{P}}{\partial z^2} - \frac{1}{2}\varepsilon\alpha \frac{B'}{B} \frac{\partial \hat{P}}{\partial z} + \varepsilon\rho\nu\alpha \frac{\partial^2 \hat{P}}{\partial z \partial \alpha} \quad \text{for } \tau > 0. \\ \hat{P} &= \delta(z) \quad \text{as } \tau \rightarrow 0. \end{aligned} \tag{F.3.38}$$

We look for ways to remove the term $\frac{1}{2}\varepsilon\alpha \frac{B'}{B} \frac{\partial \hat{P}}{\partial z}$ in equation (F.3.38). Hence, we introduce a new variable H defined by

$$\hat{P} = \left(\frac{C(f)}{C(K)} \right)^m H = \left(\frac{B(\varepsilon\alpha z)}{B(0)} \right)^m H.$$

It follows that

$$\frac{\partial \hat{P}}{\partial z} = \left(\frac{B(\varepsilon\alpha z)}{B(0)} \right)^m \left(\frac{\partial H}{\partial z} + \varepsilon m\alpha \frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} H \right) = m\varepsilon\alpha \frac{B'}{B} \hat{P} + \left(\frac{B}{B_0} \right)^m \frac{\partial H}{\partial z}.$$

Also,

$$\begin{aligned} \frac{\partial^2 \hat{P}}{\partial z^2} &= \left(\frac{B}{B_0} \right)^m \frac{\partial^2 H}{\partial z^2} + 2m\varepsilon\alpha \frac{B'}{B} \left(\frac{B}{B_0} \right)^m \frac{\partial H}{\partial z} \\ &\quad + m\varepsilon^2\alpha^2 \left(\frac{B}{B_0} \right)^m \times H \times \left(\frac{mB'^2 + B''B - B'^2}{B^2} \right). \end{aligned}$$

Lastly, we have

$$\begin{aligned} \frac{\partial^2 \hat{P}}{\partial z \partial \alpha} &= \left(\frac{B(\varepsilon\alpha z)}{B(0)} \right)^m \left[m\varepsilon \frac{B'}{B} H + m\alpha\varepsilon^2 H z \frac{B''}{B} + \alpha \frac{B'}{B^2} m(m-1)\varepsilon^2 z H \right. \\ &\quad \left. + m\varepsilon\alpha \frac{B'}{B} \frac{\partial H}{\partial \alpha} + m\varepsilon z \frac{B'}{B} \frac{\partial H}{\partial z} + \frac{\partial^2 H}{\partial z \partial \alpha} \right]. \end{aligned}$$

Hence, taking $\varepsilon^2 \rightarrow 0$, we obtain

$$\frac{\partial^2 \hat{P}}{\partial z \partial \alpha} = \left(\frac{B(\varepsilon \alpha z)}{B(0)} \right)^m \left(\frac{\partial^2 \hat{H}}{\partial z \partial \alpha} + \varepsilon m z \frac{B'}{B} \frac{\partial H}{\partial z} + \varepsilon m \alpha \frac{B'}{B} \frac{\partial H}{\partial \alpha} + \varepsilon m \frac{B'}{B} H + \mathcal{O}(\varepsilon^2) \right).$$

Consequently, the price option in equation (F.3.34) is now given by

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \alpha \varepsilon B(0) \left(\frac{B(\varepsilon \alpha z)}{B(0)} \right)^m \int_0^{\tau_{ex}} H(\tau, z, \alpha) d\tau. \quad (\text{F.3.39})$$

Substituting these results in equation (F.3.38), ignoring all the $\mathcal{O}(\varepsilon^3)$ terms and doing some elementary calculation, we get

$$\begin{aligned} \frac{\partial H}{\partial \tau} = & \frac{1}{2} (1 - 2\varepsilon \rho \nu z + \varepsilon^2 \nu^2 z^2) \frac{\partial^2 H}{\partial z^2} + \varepsilon \alpha \left(m - \frac{1}{2} \right) \frac{B'}{B} \frac{\partial H}{\partial z} \\ & + \frac{1}{2} \varepsilon^2 \alpha^2 \left[m \frac{B''}{B} + (m^2 - 2m) \frac{B'^2}{B^2} \right] H - \varepsilon^2 m \rho \nu \alpha \frac{B'}{B} \left(z \frac{\partial H}{\partial z} - H \right) \\ & + \varepsilon \rho \nu \alpha \left(\frac{\partial^2 H}{\partial z \partial \alpha} + \varepsilon m \alpha \frac{B'}{B} \frac{\partial H}{\partial \alpha} \right). \end{aligned}$$

We take $m = \frac{1}{2}$ to cancel the term $\frac{B'}{B} \frac{\partial H}{\partial z}$, consequently our option price is now

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \alpha \varepsilon B(0) \left(\frac{B(\varepsilon \alpha z)}{B(0)} \right)^{\frac{1}{2}} \int_0^{\tau_{ex}} H(\tau, z, \alpha) d\tau \quad (\text{F.3.40})$$

where $H(\tau, z, \alpha)$ is the solution of

$$\begin{aligned} \frac{\partial H}{\partial \tau} = & \frac{1}{2} (1 - 2\varepsilon \rho \nu z + \varepsilon^2 \nu^2 z^2) \frac{\partial^2 H}{\partial z^2} + \varepsilon^2 \alpha^2 \left[\frac{1}{4} \frac{B''}{B} - \frac{3}{8} \frac{B'^2}{B^2} \right] H \\ & - \frac{1}{2} \varepsilon^2 \rho \nu \alpha \frac{B'}{B} \left(z \frac{\partial H}{\partial z} - H \right) + \varepsilon \rho \nu \alpha \left(\frac{\partial^2 H}{\partial z \partial \alpha} + \varepsilon \frac{1}{2} \alpha \frac{B'}{B} \frac{\partial H}{\partial \alpha} \right). \end{aligned} \quad (\text{F.3.41})$$

for $\tau > 0$, with initial condition $H = \delta(z)$ at $\tau = 0$. Taking all the $\mathcal{O}(1)$ terms, we have the following problem

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2} \frac{\partial^2 H}{\partial z^2} \\ H &= \delta(z). \end{aligned}$$

This is an heat equation and can be solved using equation (G.0.7). The solution $H_0 = H_0(z, \tau)$. Therefore, we can expand H as

$$H(\tau, z, \alpha) = H_0(\tau, z) + \varepsilon H_1(\tau, z, \alpha) + \varepsilon^2 H_2(\tau, z, \alpha) + \dots \quad (\text{F.3.42})$$

Substituting (F.3.42) into (F.3.41), we get

$$\begin{aligned} \frac{\partial H_0}{\partial \tau} + \varepsilon \frac{\partial H_1}{\partial \tau} + \varepsilon^2 \frac{\partial H_2}{\partial \tau} + \dots &= \frac{1}{2}(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2) \left[\frac{\partial^2 H_0}{\partial z^2} + \varepsilon \frac{\partial^2 H_1}{\partial z^2} + \varepsilon^2 \frac{\partial^2 H_2}{\partial z^2} + \dots \right] \\ &+ \varepsilon^2 \alpha^2 \left[\frac{1}{4} \frac{B''}{B} - \frac{3}{8} \frac{B'^2}{B^2} \right] \left(H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \dots \right) \\ &- \frac{1}{2} \varepsilon^2 \rho\nu \alpha \frac{B'}{B} \left[z \left(\frac{\partial H_0}{\partial z} + \varepsilon \frac{\partial H_1}{\partial z} + \varepsilon^2 \frac{\partial H_2}{\partial z} + \dots \right) - \left(H_0 + \varepsilon H_1 + \dots \right) \right] \\ &+ \varepsilon \rho\nu \alpha \left[\left(\frac{\partial^2 H_0}{\partial z \partial \alpha} + \varepsilon \frac{\partial^2 H_1}{\partial z \partial \alpha} + \dots \right) + \varepsilon \frac{1}{2} \alpha \frac{B'}{B} \left(\frac{\partial H_0}{\partial \alpha} + \varepsilon \frac{\partial H_1}{\partial \alpha} + \dots \right) \right]. \end{aligned}$$

The fact that H_0 is independent of α and ignoring all the $\mathcal{O}(\varepsilon^3)$ terms, we note the following $\frac{\partial^2 H_0}{\partial z \partial \alpha} = 0$, $\frac{\partial H_0}{\partial \alpha} = 0$, $\varepsilon^2 \left(z \frac{\partial^2 H_0}{\partial z^2} - H_0 \right) = \varepsilon^2 \left(z \frac{\partial^2 H}{\partial z^2} - H \right)$, and $\varepsilon^2 H_0 = \varepsilon^2 H$. Considering all the $\mathcal{O}(\varepsilon)$ terms, we have

$$\frac{\partial H_1}{\partial \tau} = \frac{1}{2} \frac{\partial^2 H_1}{\partial z^2} - \rho\nu z \frac{\partial^2 H_0}{\partial z^2}.$$

for $\tau > 0$, with initial condition $H = \delta(z)$ at $\tau = 0$. This is a heat equation and can be solved using formula (G.0.7). The solution $H_1 = H_1(z, \tau)$. Likewise, we prove that $H_2 = H_2(z, \tau)$ and so on. Therefore, $H = H(z, \tau)$. Consequently equation (F.3.41) is reduced to

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2}(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2) \frac{\partial^2 H}{\partial z^2} + \varepsilon^2 \alpha^2 \left[\frac{1}{4} \frac{B''}{B} + \frac{3}{8} \frac{B'^2}{B^2} \right] H \\ &- \frac{1}{2} \varepsilon^2 \rho\nu \alpha \frac{B'}{B} \left(z \frac{\partial H}{\partial z} - H \right) \end{aligned} \quad (\text{F.3.43})$$

for $\tau > 0$, with initial condition $H = \delta(z)$ at $\tau = 0$. For any fixed z_0

$$\frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} = \frac{B'(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)} + \mathcal{O}(\varepsilon) \quad \text{and} \quad \frac{B''(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} = \frac{B''(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)} + \mathcal{O}(\varepsilon).$$

So up to and including $\mathcal{O}(\varepsilon^2)$, B'/B and B''/B can be replaced by any constant evaluated at any convenient z_0 . Therefore we can define the following

$$b_1 := \frac{B'(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)} \quad \text{and} \quad b_2 := \frac{B''(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)}.$$

Let us define a new variable \hat{H} by

$$H = \exp \left(\frac{1}{4} \varepsilon^2 \rho\nu \alpha b_1 z^2 \right) \hat{H}.$$

It follows that

$$\frac{\partial H}{\partial \tau} = \exp\left(\frac{1}{4}\varepsilon^2 \rho \nu \alpha b_1 z^2\right) \frac{\partial \hat{H}}{\partial \tau}$$

and

$$\frac{\partial H}{\partial z} = \exp\left(\frac{1}{4}\varepsilon^2 \rho \nu \alpha b_1 z^2\right) \left(\frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2 \rho \nu \alpha b_1 z \hat{H}\right).$$

Moreover

$$\begin{aligned} \frac{\partial^2 H}{\partial z^2} &= \frac{1}{2}\varepsilon^2 \rho \nu \alpha b_1 z \times \exp\left(\frac{1}{4}\varepsilon^2 \rho \nu \alpha b_1 z^2\right) \times \left(\frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2 \rho \nu \alpha b_1 z \hat{H}\right) \\ &\quad + \exp\left(\frac{1}{4}\varepsilon^2 \rho \nu \alpha b_1 z^2\right) \left[\frac{\partial^2 \hat{H}}{\partial z^2} + \frac{1}{2}\varepsilon^2 \rho \nu \alpha b_1 \hat{H} + \frac{1}{2}\varepsilon^2 \rho \nu \alpha b_1 z \frac{\partial \hat{H}}{\partial z}\right]. \end{aligned}$$

Ignoring the $\mathcal{O}(\varepsilon^4)$ term, we have

$$\frac{\partial^2 H}{\partial z^2} = \exp\left(\frac{1}{4}\varepsilon^2 \rho \nu \alpha b_1 z^2\right) \left(\frac{\partial^2 \hat{H}}{\partial z^2} + \varepsilon^2 \rho \nu \alpha b_1 z \frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2 \rho \nu \alpha b_1 \hat{H}\right).$$

Consequently, our option price (F.3.40) becomes

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2}\alpha\varepsilon\sqrt{B(0)B(\varepsilon\alpha z)}e^{\varepsilon^2 \rho \nu \alpha b_1 z^2/4} \int_0^{\tau_{ex}} \hat{H}(\tau, z, \alpha) d\tau.$$

In addition, substituting these results in equation (F.3.43) and ignoring all the $\mathcal{O}(\varepsilon^3)$ term, we have

$$\frac{\partial \hat{H}}{\partial \tau} = \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right) \frac{\partial^2 \hat{H}}{\partial z^2} + \frac{3}{4}\varepsilon^2 \rho \nu \alpha b_1 \hat{H} + \varepsilon^2 \alpha^2 \left[\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right] \hat{H}$$

for $\tau > 0$, with initial condition $\hat{H} = \delta(z)$ at $\tau = 0$. Afterwards, we define

$$x := \frac{1}{\varepsilon\nu} \int_0^{\varepsilon\nu z} \frac{dt}{\sqrt{1 - 2\rho t + t^2}}.$$

We compute the integral J and then substitute the limits to get x .

$$J := \int \frac{dt}{\sqrt{1 - 2\rho t + t^2}} = \int \frac{dt}{\sqrt{1 - \rho^2} \sqrt{1 + \frac{(t-\rho)^2}{1-\rho^2}}}.$$

We perform a change of variable twice, we define

$$\frac{t - \rho}{\sqrt{1 - \rho^2}} = u \quad \text{and} \quad u = \tan \theta.$$

It follows that

$$J = \int \frac{d\theta}{\cos^2 \theta \sqrt{1 + \tan^2 \theta}} = \int \frac{\cos \theta d\theta}{\cos^2 \theta} = \int \frac{\cos \theta d\theta}{1 - \sin^2 \theta} = \frac{1}{2} \log \left| \frac{1 + \sin \theta}{1 - \sin \theta} \right|.$$

Going back to the old variables above, and since

$$\sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} \quad \text{and} \quad u^2 + 1 = \frac{1 + t^2 - 2\rho t}{1 - \rho^2},$$

we have

$$J = \frac{1}{2} \log \left\{ \frac{\frac{t-\rho}{\sqrt{1-\rho^2}} + \frac{\sqrt{1+t^2-2\rho t}}{\sqrt{1-\rho^2}}}{\frac{\sqrt{1+t^2-2\rho t}}{\sqrt{1-\rho^2}} - \frac{t-\rho}{\sqrt{1-\rho^2}}} \right\} = \frac{1}{2} \log \left\{ \frac{\left(\sqrt{1+t^2-2\rho t} + (t-\rho) \right)^2}{1+t^2-2\rho t - (t^2-2\rho t+\rho^2)} \right\}.$$

It follows

$$x = \frac{1}{\varepsilon\nu} \left[\log \left(\frac{\sqrt{1+t^2-2\rho t} + t - \rho}{\sqrt{1-\rho^2}} \right) \right]_0^{\varepsilon\nu z} = \frac{1}{\varepsilon\nu} \log \left(\frac{\sqrt{1-\rho^2+y^2} + y}{1-\rho} \right).$$

where y is given by

$$y := \varepsilon\nu z - \rho.$$

For $\rho \neq 1$ we have,

$$(1-\rho)e^{2\varepsilon\nu x} - 2ye^{\varepsilon\nu x} = 1+\rho \implies y = \frac{1}{2} \left[(e^{\varepsilon\nu x} - e^{-\varepsilon\nu x}) - \rho(e^{\varepsilon\nu x} + e^{-\varepsilon\nu x}) \right].$$

Therefore

$$y = \sinh(\varepsilon\nu x) - \rho \cosh(\varepsilon\nu x) \implies \varepsilon\nu z = \sinh(\varepsilon\nu x) - \rho(\cosh(\varepsilon\nu x) - 1).$$

After the transformation from z to x , we have

$$\frac{\partial \hat{H}}{\partial z} = \frac{\partial x}{\partial z} \frac{\partial \hat{H}}{\partial x} = \frac{1}{\sqrt{1-2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2}} \frac{\partial \hat{H}}{\partial x} = \frac{1}{I(\varepsilon\nu z)} \frac{\partial \hat{H}}{\partial x}.$$

Also

$$\frac{\partial^2 \hat{H}}{\partial z^2} = -\varepsilon\nu \frac{I'}{I^2} \frac{\partial \hat{H}}{\partial x} + \frac{1}{I^2} \frac{\partial^2 \hat{H}}{\partial x^2}.$$

where, I is defined by

$$I(t) = \sqrt{1-2\rho t + t^2}.$$

In terms of x , we have

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \alpha \varepsilon \sqrt{B(0)B(\varepsilon\alpha z)} e^{\varepsilon^2 \rho \nu \alpha b_1 z^2 / 4} \int_0^{\tau_{ex}} \hat{H}(\tau, x) d\tau$$

and

$$\frac{\partial \hat{H}}{\partial \tau} = \frac{1}{2} \frac{\partial^2 \hat{H}}{\partial x^2} - \frac{1}{2} \varepsilon \nu I'(\varepsilon \nu z) \frac{\partial \hat{H}}{\partial x} + \frac{3}{4} \varepsilon^2 \rho \nu \alpha b_1 \hat{H} + \varepsilon^2 \alpha^2 \left[\frac{1}{4} b_2 - \frac{3}{8} b_1^2 \right] \hat{H}$$

for $\tau > 0$, with initial condition $\hat{H} = \delta(x)$ at $\tau = 0$. Lastly, we define $Q(\tau, x)$ by

$$\hat{H} = I^{\frac{1}{2}}(\varepsilon \nu z(x)) Q = \left(1 - 2\varepsilon \rho \nu z + \varepsilon^2 \nu^2 z^2\right)^{\frac{1}{4}} Q.$$

It follows

$$\frac{\partial \hat{H}}{\partial \tau} = I^{\frac{1}{2}}(\varepsilon \nu z) \frac{\partial Q}{\partial \tau} \quad \text{and} \quad \frac{\partial \hat{H}}{\partial x} = Q \frac{\partial z}{\partial x} \frac{\partial I^{\frac{1}{2}}}{\partial z} + I^{\frac{1}{2}} \frac{\partial Q}{\partial x}.$$

Finally

$$\begin{aligned} \frac{\partial^2 \hat{H}}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial \hat{H}}{\partial x} \right) = \frac{\partial z}{\partial x} \frac{1}{2} \varepsilon \nu I' I^{-\frac{1}{2}} \left(\frac{\partial Q}{\partial x} + \frac{1}{2} \varepsilon \nu I'(\varepsilon \nu z) Q \right) \\ &\quad + I^{\frac{1}{2}}(\varepsilon \nu z) \left[\frac{\partial^2 Q}{\partial x^2} + \frac{1}{2} \varepsilon \nu \left(\varepsilon \nu I \frac{\partial^2 I}{\partial z^2} Q + I'(\varepsilon \nu z) \frac{\partial Q}{\partial x} \right) \right]. \end{aligned}$$

Consequently, our option price is now

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \alpha \varepsilon \sqrt{B(0)B(\varepsilon \alpha z)} I^{\frac{1}{2}}(\varepsilon \nu z) e^{\frac{1}{4} \varepsilon^2 \rho \nu \alpha b_1 z^2} \int_0^{\tau_{ex}} Q(\tau, x) d\tau$$

with Q solution of

$$\frac{\partial Q}{\partial \tau} = \frac{1}{2} \frac{\partial^2 Q}{\partial x^2} + \varepsilon^2 \nu^2 \left(\frac{1}{4} I'' I - \frac{1}{8} I'^2 \right) Q + \frac{3}{4} \varepsilon^2 \rho \nu \alpha b_1 Q + \varepsilon^2 \alpha^2 \left[\frac{1}{4} b_2 - \frac{3}{8} b_1^2 \right] Q$$

for $\tau > 0$, with initial condition $Q = \delta(x)$ at $\tau = 0$. We can substitute $I(\varepsilon \nu z)$, $I'(\varepsilon \nu z)$, $I''(\varepsilon \nu z)$ by the constants $I(\varepsilon \nu z_0)$, $I'(\varepsilon \nu z_0)$, $I''(\varepsilon \nu z_0)$ and commit only $\mathcal{O}(\varepsilon)$ errors. We will choose z_0 later on. We define the constant κ by

$$\kappa = \nu^2 \left(\frac{1}{4} I''(\varepsilon \nu z_0) I(\varepsilon \nu z_0) - \frac{1}{8} [I'(\varepsilon \nu z_0)]^2 \right) + \frac{3}{4} \rho \nu \alpha b_1 + \alpha^2 \left[\frac{1}{4} b_2 - \frac{3}{8} b_1^2 \right].$$

Therefore through $\mathcal{O}(\varepsilon^2)$, we can simplify our problem to

$$\begin{aligned} \frac{\partial Q}{\partial \tau} &= \frac{1}{2} \frac{\partial^2 Q}{\partial x^2} + \varepsilon^2 \kappa Q \quad \text{for } \tau > 0, \\ Q &= \delta(x) \quad \text{at } \tau = 0. \end{aligned}$$

This is an heat equation and can be solved using (G.0.7). The solution is

$$Q(\tau, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} e^{\varepsilon^2 \kappa \tau}.$$

We observe that the Taylor series of $e^{\varepsilon^2 \kappa \tau}$ at $\varepsilon = 0$ is the same as the Taylor series of $\frac{1}{(1 - \frac{2}{3} \kappa \varepsilon^2 \tau + \dots)^{\frac{3}{2}}}$ at $\varepsilon = 0$. Hence, through (ε^2) , we have

$$Q(\tau, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} \frac{1}{\left(1 - \frac{2}{3} \kappa \varepsilon^2 \tau + \dots\right)^{\frac{3}{2}}}.$$

The option price is given by

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \frac{f - K}{x} \int_0^{\tau_{ex}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} e^{\varepsilon^2 \theta} e^{\varepsilon^2 \kappa \tau} d\tau, \quad (\text{F.3.44})$$

where we have defined $\varepsilon^2 \theta$ by

$$\varepsilon^2 \theta := \log \left(\frac{\varepsilon \alpha z}{f - K} \sqrt{B(0)B(\varepsilon \alpha z)} \right) + \log \left(\frac{x I^{\frac{1}{2}}(\varepsilon \nu z)}{z} \right) + \frac{1}{4} \varepsilon^2 \rho \nu \alpha b_1 z^2.$$

Expanding $\varepsilon^2 \theta$ through ε^2 yields

$$\begin{aligned} \varepsilon^2 \theta \sim & \left(\frac{1}{12} b_2 - \frac{1}{8} b_1^2 \right) \varepsilon^2 \alpha^2 z^2 + \left[\frac{1}{12} \frac{I''(\frac{1}{2} \varepsilon \nu z)}{I(\frac{1}{2} \varepsilon \nu z)} - \frac{1}{24} \left(\frac{I'(\frac{1}{2} \varepsilon \nu z)}{I_0(\frac{1}{2} \varepsilon \nu z)} \right)^2 \right] \varepsilon^2 \nu^2 z^2 \\ & + \frac{1}{4} \varepsilon^2 \rho \nu \alpha b_1 z^2. \end{aligned}$$

Using the fact

$$\frac{z}{x} = 1 + \mathcal{O}(\varepsilon) \quad \text{and} \quad I(\varepsilon \nu z) = 1 + \dots,$$

we note that $\frac{\theta}{x^2}$ matches

$$\frac{\kappa}{3} = \frac{1}{3} \nu^2 \left(\frac{1}{4} I''(\varepsilon \nu z_0) I(\varepsilon \nu z_0) - \frac{1}{8} [I'(\varepsilon \nu z_0)]^2 \right) + \frac{1}{4} \rho \nu \alpha b_1 + \frac{1}{3} \alpha^2 \left[\frac{1}{4} b_2 - \frac{3}{8} b_1^2 \right]$$

and therefore

$$e^{\varepsilon^2 \kappa \tau} = \frac{1}{\left(1 - \frac{2}{3} \kappa \varepsilon^2 \tau + \dots\right)^{\frac{3}{2}}} = \frac{1}{\left(1 - 2\varepsilon^2 \tau \frac{\theta}{x^2}\right)^{\frac{3}{2}}} + \mathcal{O}(\varepsilon^4)$$

through $\mathcal{O}(\varepsilon^2)$. Hence, our option price becomes

$$V(t, f, \alpha) = (f - K)^+ + \frac{1}{2} \frac{f - K}{x} \int_0^{\tau_{ex}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} e^{\varepsilon^2 \theta} \frac{1}{\left(1 - 2\varepsilon^2 \tau \frac{\theta}{x^2}\right)^{\frac{3}{2}}} d\tau.$$

Changing integration variables to $q := \frac{x^2}{2\tau}$, we get

$$V(t, f, \alpha) = (f - K)^+ + \frac{|f - K|}{4\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{ex}}}^{\infty} \frac{e^{-q + \varepsilon^2 \theta}}{\left(q - \varepsilon^2 \theta\right)^{\frac{3}{2}}} dq.$$

It follows that, the value of a European call option is given by

$$V(t, f, \alpha) = (f - K)^+ + \frac{|f - K|}{4\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{ex}} - \varepsilon^2\theta}^{\infty} q^{-\frac{3}{2}} e^{-q} dq, \quad (\text{F.3.45})$$

with⁵

$$\varepsilon^2\theta = \log \left(\frac{\varepsilon\alpha z}{f - K} \sqrt{B(0)B(\varepsilon\alpha z)} \right) + \log \left(\frac{xI^{\frac{1}{2}}(\varepsilon\nu z)}{z} \right) + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2$$

through $\mathcal{O}(\varepsilon^2)$.

⁵This is found in (Hagan *et al.*, 2002).

Appendix G

Fourier transform

We give some useful formulae for the Fourier transform, which provide powerful tools to solve the heat equation. Here, for a given complex function f , \bar{f} denotes its complex conjugate.

G.0.1 Definitions and elementary properties

Definition G.1 (Fourier transform on L^1). *If $u \in L^1(\mathbb{R}^n)$, we define its Fourier transform*

$$\hat{u}(y) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot y} u(x) dx \quad (y \in \mathbb{R}^n) \quad (\text{G.0.1})$$

and its inverse Fourier transform

$$u^\sim(y) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot y} u(x) dx \quad (y \in \mathbb{R}^n) \quad (\text{G.0.2})$$

Theorem G.2 (Properties of Fourier transform). *Assume $u, v \in L^2(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} u \bar{v} dx = \int_{\mathbb{R}^n} \hat{u} \bar{\hat{v}} dy; \quad (\text{G.0.3})$$

$$\widehat{D^\alpha u} = (iy)^\alpha \hat{u} \quad \text{for each multiindex } \alpha \text{ such that } D^\alpha u \in L^2(\mathbb{R}^n); \quad (\text{G.0.4})$$

$$(u \hat{*} v) = (2\pi)^{n/2} \hat{u} \hat{v}; \quad (\text{G.0.5})$$

$$u = \hat{\hat{u}}^\sim. \quad (\text{G.0.6})$$

G.0.2 Inhomogeneous Heat Equation by Fourier transform

We consider the initial value problem for $u = u(x, t)$, $u \in C^\infty(\mathbb{R}^n \times R_+)$,

$$\begin{aligned} u_t - \Delta u &= f(x, t) \quad \text{in } x \in \mathbb{R}^n, t \geq 0, \\ u(x, 0) &= \phi(x), \end{aligned}$$

where ϕ and f are given. By Fourier transform, we have

$$\begin{aligned} \widehat{u_t - \Delta u} &= \widehat{f(x, t)} \\ \widehat{u(x, 0)} &= \widehat{\phi(x)}. \end{aligned}$$

We obtain an initial value problem for an ordinary differential equation:

$$\begin{aligned} \frac{d\hat{u}}{dt} + |\xi|^2 \hat{u} &= \hat{f}(\xi, t) \\ \hat{u}(\xi, 0) &= \hat{\phi}(\xi). \end{aligned}$$

The solution is given by

$$\hat{u}(\xi, t) = e^{-|\xi|^2 t} \hat{\phi}(\xi) + \int_0^t e^{-|\xi|^2(t-\tau)} \hat{f}(\xi, \tau) d\tau.$$

Using the inverse Fourier transform, we have

$$u(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left(e^{-|\xi|^2 t} \hat{\phi}(\xi) + \int_0^t e^{-|\xi|^2(t-\tau)} \hat{f}(\xi, \tau) d\tau \right) d\xi.$$

After calculation, we have

$$\begin{aligned} u(x, t) &= \frac{1}{(2\sqrt{\pi t})^n} \int_{\mathbb{R}^n} \phi(y) e^{-|x-y|^2/4t} dy \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(2\sqrt{\pi(t-\tau)})^n} f(y, \tau) e^{-|x-y|^2/4(t-\tau)} dy d\tau. \end{aligned} \quad (\text{G.0.7})$$

Appendix H

Computer code

In this appendix, we give a summary of some the computer codes. Should you have questions or concerns regarding codes, please feel free to drop me a message or email at jeffrey@aims.ac.za or ted.mbongo@ymail.com.

H.1 SABR model

It is here where we give a summary of the steps taken to implement the SABR model. This rather a user guide which could be used in conjunction with the code to understand our implementation. The implementation was done in Matlab. We use functions written by of **Fabrice Douglas Rouah** (available at <http://www.volopta.com>).

This function estimates all the parameters simultaneously using Method 1

```
function y = EstimateAllParameters(params,MktStrike,MktVol,F,T,b)
% -----
% Returns the following SABR parameters:
% a = alpha
% r = rho
% v = vol-of-vol
% Required inputs:
% MktStrike = Vector of Strikes
% MktVol     = Vector of corresponding volatilities
% F = spot price
% T = maturity
% b = beta parameter
% -----
a = params(1);
r = params(2);
v = params(3);
N = length(MktVol);
% Define the model volatility and the squared error terms
```

```

for i=1:N
ModelVol(i) = SABRvol(a,b,r,v,F,MktStrike(i),T);
error(i) = (ModelVol(i) - MktVol(i))^2;
end;
% Return the SSE
y = sum(error);
% Impose the constraint that -1 <= rho <= +1 and that v>0
if abs(r)>1 | v<0
y = 1e100;
end

```

This function estimates all the parameters simultaneously using Method 2

```

function y = EstimateAllParametersFine_Tune(params,
    MktStrike,MktVol,F,T,b,FineTune)
% -----
% Uses the Fine Tuned version of the SABR volatilities.
% Returns the following SABR parameters:
% a = alpha
% r = rho
% v = vol-of-vol
% Required inputs:
% MktStrike = Vector of Strikes
% MktVol     = Vector of corresponding volatilities
% F = spot price
% T = maturity
% b = beta parameter
% FineTune? ('Y' or 'N')
% -----
a = params(1);
r = params(2);
v = params(3);
N = length(MktVol);
% Define the model volatility and the squared error terms
for i=1:N
ModelVol(i) = SABRvol_FineTune(a,b,r,v,F,MktStrike(i),T,FineTune);
error(i) = (ModelVol(i) - MktVol(i))^2;
end;
% Return the SSE
y = sum(error);
% Impose the constraint that -1 <= rho <= +1 and that v>0
if abs(r)>1 | v<0
y = 1e100;
end

```

This function estimates the parameters ρ and ν simultaneously using Method 1

```
function y = EstimateRhoAndVol(params,MktStrike,MktVol,ATMVol,F,T,b)
% -----
% Returns the following SABR parameters:
% r = rho
% v = vol-of-vol
% Uses ATM volatility to estimate alpha
% Required inputs:
% MktStrike = Vector of Strikes
% MktVol     = Vector of corresponding volatilities
% ATMVol = ATM volatility
% F = spot price
% T = maturity
% b = beta parameter
% -----
r = params(1);
v = params(2);
a = findAlpha(F,F,T,ATMVol,b,r,v);
N = length(MktVol);
% Define the model volatility and the squared error terms
for i=1:N
ModelVol(i) = SABRvol(a,b,r,v,F,MktStrike(i),T);
error(i) = (ModelVol(i) - MktVol(i))^2;
end;
% Return the SSE
y = sum(error);
% Impose the constraint that -1 <= rho <= +1 and that v>0
% via a penalty on the objective function
if abs(r)>1 | v<0
y = 1e100;
end
```

This function estimates the parameters ρ and ν simultaneously using Method 2

```
function y = EstimateRhoAndVol(params,
    MktStrike,MktVol,ATMVol,F,T,b,FineTune)
% -----
% Uses the Fine Tuned version of the SABR volatilities.
% Returns the following SABR parameters:
% r = rho
% v = vol-of-vol
% Uses ATM volatility to estimate alpha
```

```

% Required inputs:
% MktStrike = Vector of Strikes
% MktVol     = Vector of corresponding volatilities
% ATMVol     = ATM volatility
% F = spot price
% T = maturity
% b = beta parameter
% Fine Tune? 'Y' or 'N'
% -----
r = params(1);
v = params(2);
a = findAlpha(F,F,T,ATMVol,b,r,v);
N = length(MktVol);
% Define the model volatility and the squared error terms
for i=1:N
ModelVol(i)=SABRvol_FineTune(a,b,r,v,F,MktStrike(i),T,FineTune);
error(i) = (ModelVol(i) - MktVol(i))^2;
end;
% Return the SSE
y = sum(error);
% Impose the constraint that -1 <= rho <= +1 and that v>0
% via a penalty on the objective function
if abs(r)>1 | v<0
y = 1e100;
end

```

This function returns the SABR volatility

```

function y = SABRvol(a,b,r,v,F,K,T);
% -----
% Returns the SABR volatility.
% Required inputs:
% a = alpha parameter
% b = beta parameter
% r = rho parameter
% v = vol of vol parameter
% F = spot price
% K = strike price
% T = maturity
% -----
if abs(F-K) <= 0.001 % ATM vol
Term1 = a/F^(1-b);
Term2 = ((1-b)^2/24*a^2/F^(2-2*b) +
r*b*a*v/4/F^(1-b) + (2-3*r^2)*v^2/24);

```



```

y = Term1*(1 + Term2*T);
else % Non-ATM vol
FK = F*K;
z = v/a*(FK)^((1-b)/2)*log(F/K);
x = log((sqrt(1 - 2*r*z + z^2) + z - r)/(1-r));
Term1 = a / FK^((1-b)/2) / (1 + (1-b)^2/24*log(F/K)^2 +
(1-b)^4/1920*log(F/K)^4);
if abs(x-z) < 1e-10
Term2 = 1;
else
Term2 = z / x;
end
Term3 = 1 + ((1-b)^2/24*a^2/FK^(1-b) + r*b*v*a/4/FK^((1-b)/2) +
(2-3*r^2)/24*v^2)*T;
y = Term1*Term2*Term3;
end

```

This function returns the SABR volatility using Method 2

```

function y = SABRvol_FineTune(a,b,r,v,F,K,T,FineTune);
% -----
% Returns two possible SABR volatilities.
% (1) Original SABR formula by Hagan et al.
% (2) Fine Tuned formula summarized in
%     "Fine-Tune Your Smile" by Jan Obloj
% Required inputs:
% a = alpha parameter
% b = beta parameter
% r = rho parameter
% v = vol of vol parameter
% F = spot price
% K = strike price
% T = maturity
% FineTune? ('Y' or 'N')
% -----
x = log(F/K);
% Separate out into cases x=0, b=1, and b<1.
if abs(x) <= 0.0001
IOH = a*K^(b-1);
IOB = a*K^(b-1);
else
if abs(v) <= 0.001 % ATM vol
IOH = x*a*(1-b) / (F^(1-b)-K^(1-b));
IOB = x*a*(1-b) / (F^(1-b)-K^(1-b));

```

```

else
if b == 1
z = v*x/a;
sq = sqrt(1 - 2*r*z + z^2);
IOH = v*x / log((sq + z - r)/(1-r));
IOB = v*x / log((sq + z - r)/(1-r));
else
z = v*(F^(1-b) - K^(1-b)) / a / (1-b);
e = v*(F-K) / a / (F*K)^(b/2);
sq = sqrt(1-2*r*e + e^2);
IOH = v*x*e / z / log((sq + e - r)/(1-r));
IOB = v*x / log((sq + z - r)/(1-r));
end
end
end
I1H = (b-1)^2*a^2/24/(F*K)^(1-b) + r*v*a*b/4/(F*K)^((1-b)/2) +
(2-3*r^2)*v^2/24;
% Original Hagan SABR implied vol.
HaganVol = IOH*(1 + I1H*T);
% Fine Tuned SABR implied vol.
RefinedVol = IOB*(1 + I1H*T);
if strcmp(FineTune,'N')
y = HaganVol;
else
y = RefinedVol;
end

```

This function estimates the parameter α

```

function y = findAlpha(F,K,T,ATMvol,b,r,v)
% -----
% Required inputs:
% F = spot price
% K = strike price
% T = maturity
% ATMvol = ATM market volatility
% b = beta parameter
% r = rho parameter
% v = vol of vol parameter
% -----
% Find the coefficients of the cubic equation for alpha
C0 = -ATMvol*F^(1-b);
C1 = (1 + (2-3*r^2)*v^2*T/24);
C2 = r*b*v*T/4/F^(1-b);

```

```

C3 = (1-b)^2*T/24/F^(2-2*b);
% Return the roots of the cubic equation (multiple roots)
AlphaVector = roots([C3 C2 C1 C0]);
% Find and return the smallest positive root
index = find(AlphaVector>0);
Alpha = AlphaVector(index);
y = min(Alpha);

```

This is a short section from the code for calibrating a 5Y12Y swaption, given the above functions.

```

%% Calibrating the smile for a 5Y12Y swaption
% Define the starting values and options for fminsearch
start = [0.3,0.3];
options = optimset('MaxFunEvals', 1e5, 'TolFun',
1e-8, 'TolX', 1e-10);
% Parameter estimation method 1. Set Beta = 0.5.
% Estimate rho and v, and at each iteration step,
% Find alpha as the cubic root using the findAlpha function
Beta = 0.5;
[param, feval] = fminsearch(@(par)EstimateRhoAndVol
(par,K12Y,Vol5Y(:,8),Vol12ATM(8),Swapcurve(12),Expiry(8)/12,
Beta), start,options);
R1 = param(1);
V1 = param(2);
A1 = findAlpha(Swapcurve(12),Swapcurve(12),Expiry(8)/12,
Vol12ATM(8), Beta,R1,V1);
for j=1:length(K12Y);
    SABRVol5Y12Y_method1(j) = SABRVol(A1,Beta,R1,V1,
    Swapcurve(12), K12Y(j),Expiry(8)/12);
end

% Parameter estimation method 2. Set beta = 0.5.
% Estimate rho, v, and alpha directly
Beta = 0.5;
start = [0.3,0.3,0.3];
[params, feval] = fminsearch(@(par)EstimateAllParameters
(par,K12Y, Vol5Y(:,8), Swapcurve(12),Expiry(8)/12,Beta),
start, options);
A2 = params(1);
R2 = params(2);
V2 = params(3);
for j=1:length(K12Y);
    SABRVol5Y12Y_method2(j)=SABRVol(A2,Beta,R2,V2,Swapcurve(12),

```

```

        K12Y(j),Expiry(8)/12);
end

% Fined Tuned parameter estimation method 1.
Beta = 0.5;
start = [0.3,0.3];
FineTune = 'Y';
[param1F, feval] = fminsearch(@(par)EstimateRhoAndVol_FineTune
(par,K12Y,Vol5Y(:,8), Vol12ATM(8),Swapcurve(12),Expiry(8)/12,Beta,
FineTune), start, options);
R1F = param1F(1);
V1F = param1F(2);
A1F = findAlpha(Swapcurve(12),Swapcurve(12),Expiry(8)/12,
Vol12ATM(8),Beta,R1F,V1F);
for j=1:length(K12Y);
    SABRVol5Y12Y_method1FINE(j) = SABRVol(A1F,Beta,R1F,V1F,
    Swapcurve(12),K12Y(j),Expiry(8)/12);
end

% Fined Tuned parameter estimation method 2.
Beta = 0.5;
start = [0.3,0.3,0.3];
FineTune = 'Y';
[param2F, feval]=fminsearch(@(par)EstimateAllParameters_FineTune
(par,K12Y,Vol5Y(:,8),Swapcurve(12),Expiry(8)/12,Beta,FineTune),
start, options);
A2F = param2F(1);
R2F = param2F(2);
V2F = param2F(3);
for j=1:length(K12Y);
    SABRVol5Y12Y_method2FINE(j) = SABRVol(A2F,Beta,R2F,V2F,
    Swapcurve(12),K12Y(j),Expiry(8)/12);
end

```

H.2 Yield curves via Quantlib

It is here where we give a summary of the steps taken to implement the yield curves in C++ language via the Quantlib framework. The full code can be found at [Mbongo \(2014\)](#).

H.2.1 Eonia yield curve

```
Real eoniaquotes[] = {...}
```

```

boost::shared_ptr<RateHelper> rhon =
boost::make_shared<DepositRateHelper>(eoniaquotes[0], 1 * Days,
0, TARGET(), Following, false, Actual360());
boost::shared_ptr<OvernightIndex> eoniaBt =
boost::make_shared<Eonia>();
boost::shared_ptr<RateHelper> rhe1w =
boost::make_shared<OISRateHelper>(2, 1 * Weeks, Handle<Quote>
(boost::make_shared<SimpleQuote>(eoniaquotes[1])), eoniaBt);
boost::shared_ptr<RateHelper> rhe30y =
boost::make_shared<OISRateHelper>(2, 30 * Years, Handle<Quote>
(boost::make_shared<SimpleQuote>(eoniaquotes[33])), eoniaBt);
boost::shared_ptr<RateHelper> rhfeb13 =
boost::make_shared<DatedOISRateHelper>(
Date(16, January, 2013), Date(13, February, 2013),
Handle<Quote>(boost::make_shared<SimpleQuote>(eoniaquotes[34])),
eoniaBt);

std::vector<boost::shared_ptr<RateHelper> > eoniaratehelpers;
eoniaratehelpers += rhon, rhe1w, ..., rhe1m,
rhfeb13, rhemar13, rheapr13, rhemay13, rhejun13, rhejul13,
/*rhe2m, ..., rhe7m,*/ // replaced by forwards
rhe8m, ..., rhe21m, rhe2y, ..., rhe30y;

PiecewiseYieldCurve<Discount, LogLinear> curve(0, TARGET(),
eoniaratehelpers, Actual365Fixed(), tolerance);
curve.recalculate();
std::vector<Rate> spotRates, discountFactors, forwardRates;

Period forwardTenor(3, Months); // 3 months forward
// Construct spot curve, forward curve and discount curve
for(Size i=0; i<eoniaratehelpers.size(); i++) {
Rate rate;
Date d = eoniaratehelpers[i]->latestDate();
// spot rate
rate=curve.zeroRate(d, Actual365Fixed(), Continuous, Annual, true);
spotRates.push_back(rate);

// forward rate
rate = curve.forwardRate(d, TARGET().advance(d, forwardTenor),
dayCounterOIS, Simple, Annual, true);
forwardRates.push_back(rate);

// discount rate
rate = curve.discount(d, true);

```

```
discountFactors.push_back(rate);}

```

If we insert the forwards, we have to remove the swaps with the same maturity, or the curve will complain. For instance, if we use EONECBFEB13, remove the 2-months spot.

Building the Euribor3M yield curve with the Eonia curve as discount

MARKET DATA

```
// deposits
Rate dONDQuote=...,..., d1wQuote=..., ..., d12mQuote=...;
// FRAs Underlying Euribor3M, Euribor6M, Euribor12M
Rate Tod3MQuote=...,..., fra6x9Quote=...; Tom6MQuote=...,...,
fra18x24Quote=...; fra12x24Quote=...;
// FRAs Underlying Euribor6M(IMM)
Real IMMF3FRAQuote=...,...,IMMJ3FRAQuote=...;
// futures Underlying Euribor3M
Real FUT3MZ2Quote=..., FUT3MF3Quote=..., FUT3MG3Quote=...,
FUT3MH3Quote=..., FUT3MM3Quote=..., FUT3MU3Quote=...;
// EUR IRS on Euribor3M
Rate AB3E1YQuote=..., AB3E15MQuote=..., ....AB3E60YQuote=...;

```

We need to build an OIS discount curve that, we will use to bootstrap a forward curve (this was done in [H.2.1](#)).

```
boost::shared_ptr<YieldTermStructure> oisDiscountCurve(
    new PiecewiseYieldCurve<Discount,LogLinear>(
        settlementDate, eoniaratehelpers, termStructureDayCounter,
        tolerance));

```

QUOTES

```
// deposits
boost::shared_ptr<Quote> dONDRate(new SimpleQuote(dONDQuote));
boost::shared_ptr<Quote> d1wRate(new SimpleQuote(d1wQuote));
boost::shared_ptr<Quote> d12mRate(new SimpleQuote(d2mQuote));

// FRAs Underlying Euribor3M, Euribor6M, Euribor12M
boost::shared_ptr<Quote> Tod3MRate(new SimpleQuote(Tod3MQuote));
boost::shared_ptr<Quote> fra6x9Rate(new SimpleQuote(fra6x9Quote));
boost::shared_ptr<Quote>
fra18x24Rate(new SimpleQuote(fra18x24Quote));
boost::shared_ptr<Quote>

```

```

fra12x24Rate(new SimpleQuote(fra12x24Quote));

    // FRAs Underlying Euribor3M(IMM)
boost::shared_ptr<Quote>
IMMF3FRAPrice(new SimpleQuote(IMMF3FRAQuote));

    // Futures
boost::shared_ptr<Quote>
FUT3MZ2Price(new SimpleQuote(FUT3MZ2Quote));
    // EUR IRS on Euribor3M
boost::shared_ptr<Quote>AB3E1YRate(new SimpleQuote(AB3E1YQuote));
boost::shared_ptr<Quote>AB3E50YRate(new SimpleQuote(AB3E50YQuote));

    RATE HELPERS
// RateHelpers are built from the above quotes together with
// other instrument dependant infos. Quotes are passed in
// relinkable handles which could be relinked to some other
// data source later.

boost::shared_ptr<RateHelper> d3w(new DepositRateHelper(
    Handle<Quote>(d3wRate), 3*Weeks, fixingDays,
    calendar, Following, false, depositDayCounter));
// FRAs Underlying Euribor3M, Euribor6M, Euribor12M
boost::shared_ptr<RateHelper> fra1x4(new FraRateHelper(
    Handle<Quote>(fra1x4Rate),
    1, 4, fixingDays, calendar, ModifiedFollowing,
    true, depositDayCounter));
boost::shared_ptr<RateHelper> fra2x8(new FraRateHelper(
    Handle<Quote>(fra2x8Rate),
    2, 8, fixingDays, calendar, ModifiedFollowing,
    true, depositDayCounter));
boost::shared_ptr<RateHelper> fra12x24(new FraRateHelper(
    Handle<Quote>(fra12x24Rate),
    12, 24, fixingDays, calendar, ModifiedFollowing,
    true, depositDayCounter));

    // setup FRAs Underlying Euribor6M(IMM)
Integer FRAMonths = 3;
Date imm1 = IMM::nextDate(settlementDate);
boost::shared_ptr<RateHelper> IMMF3(new FuturesRateHelper(
    Handle<Quote>(IMMF3FRAPrice), imm1, FRAMonths, calendar,
    ModifiedFollowing, true, depositDayCounter));
imm1 = IMM::nextDate(imm1+1);
boost::shared_ptr<RateHelper> IMMG3(new FuturesRateHelper(

```

```

    Handle<Quote>(IMMG3FRAPrice), imm1, FRAMonths, calendar,
    ModifiedFollowing, true, depositDayCounter));

    // setup futures
    Integer futMonths = 3;
    Date imm = IMM::nextDate(settlementDate);
    boost::shared_ptr<RateHelper> FUT3MZ2(new FuturesRateHelper(
    Handle<Quote>(FUT3MZ2Price), imm, futMonths, calendar,
    ModifiedFollowing, true, depositDayCounter));
    imm = IMM::nextDate(imm+1);
    boost::shared_ptr<RateHelper> FUT3MF3(new FuturesRateHelper(
    Handle<Quote>(FUT3MF3Price), imm, futMonths, calendar,
    ModifiedFollowing, true, depositDayCounter));

    // setup swaps Euribor3M
    Frequency swFixedLegFrequency3 = Annual;
    BusinessDayConvention swFixedLegConvention3 = Unadjusted;
    DayCounter swFixedLegDayCounter3=Thirty360(Thirty360::European);
    boost::shared_ptr<IborIndex> swFloatingLegIndex3(new Euribor3M);

    boost::shared_ptr<RateHelper> AB3E1Y(new
    SwapRateHelper(Handle<Quote>(AB3E1YRate), 1*Years,
    calendar, swFixedLegFrequency3, swFixedLegConvention3,
    swFixedLegDayCounter3, swFloatingLegIndex3,
    Handle<Quote>(), 0 *Days,
    Handle<YieldTermStructure>(oisDiscountCurve)));

CURVE BUILDING
// Any DayCounter would be fine.
//ActualActual(AA)::ISDA ensures that 30 years is 30.0
//DayCounter termStructureDayCounter= AA(AA::ISDA);
double tolerance = 1.0e-15;

// A depo-FRA-futures-swap (on Euribor3M) curve
std::vector<boost::shared_ptr<RateHelper> >
depoFRAFutSwapInstruments;
depoFRAFutSwapInstruments.push_back(d2w);
depoFRAFutSwapInstruments.push_back(FUT3MZ2);
depoFRAFutSwapInstruments.push_back(FUT3MH3);
depoFRAFutSwapInstruments.push_back(AB3E30Y);
boost::shared_ptr<YieldTermStructure>depoFRAFutSwapTermStructure(
new PiecewiseYieldCurve<Discount,LogLinear>(settlementDate,
depoFRAFutSwapInstruments, termStructureDayCounter, tolerance));

```


List of References

- Ametrano, F. and Bianchetti, M. (2009). Bootstrapping the Illiquidity: Multiple Yield Curves Construction for Market Coherent Forward Rates Estimation. In “*Modeling Interest Rates: Advances for Derivatives Pricing*”, edited by F. Mercurio, *Risk Books*.
- Ametrano, F.M. (2011). Rates curves for forward Euribor estimation and CSA-discounting. *QuantLib Forum, London, 18 Jan. 2011*.
- Ametrano, F.M. and Bianchetti, M. (2013). Everything You Always Wanted to Know About Multiple Interest Rate Curve Bootstrapping But Were Afraid to Ask. Available at: <http://ssrn.com/abstract=2219548>.
- Andersen, L. (2007). Discount curve construction with tension splines. *Review of Derivatives Research*, vol. 10, no. 3, pp. 227–267.
- Andersen, L. and Andreasen, J. (2000). Jump-Diffusion Processes: Volatility Smile Fitting and Numerical Methods for Option Pricing. *Review of Derivatives Research*, vol. 4, no. 3, pp. 231–262.
- Bates, D.S. (1996). Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options. *Review of financial studies*, vol. 9, no. 1, pp. 69–107.
- Benaim, S., Friz, P. and Lee, R. (2008). On Black-Scholes Implied Volatility at Extreme Strikes. *Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling*, pp. 19–45.
- Berestycki, H., Busca, J. and Florent, I. (2002). Asymptotics and calibration of local volatility models. *Quantitative finance*, vol. 2, no. 1, pp. 61–69.
- Berestycki, H., Busca, J. and Florent, I. (2004). Computing the implied volatility in stochastic volatility models. *Communications on Pure and Applied Mathematics*, vol. 57, no. 10, pp. 1352–1373.
- Bianchetti, M. (2008). Two Curves, One Price: Pricing & Hedging Interest Rate Derivatives Decoupling Forwarding and Discounting Yield Curves. *Risk Magazine*.
- Bianchetti, M. and Carlicchi, M. (2011). Interest Rates After The Credit Crunch: Multiple-Curve Vanilla Derivatives and SABR. *The Capco Journal of Financial Transformation, Applied Finance n. 32, September*.

- Black, F., Derman, E. and Toy, W. (1990). A One-Factor Model of Interest Rates and its Application to Treasury Bond Options. *Financial analysts journal*, pp. 33–39.
- Black, F. and Karasinski, P. (1991). Bond and Option Pricing When Short Rates are Lognormal. *Financial Analysts Journal*, pp. 52–59.
- Bulirsch, R. and Stoer, J. (2002). *Introduction to Numerical Analysis*. Springer Heidelberg.
- Carr, P., Geman, H., Madan, D.B. and Yor, M. (2004). From Local Volatility to Local Lévy Models. *Quantitative Finance*, vol. 4, no. 5, pp. 581–588.
- Carr, P. and Wu, L. (2007). Stochastic Skew in Currency Options. *Journal of Financial Economics*, vol. 86, no. 1, pp. 213–247.
- Castagna, A. (2012). Pricing of Collateralized Derivatives Contracts When More than One Currency are Involved: Liquidity and Funding Value Adjustments. Available at: <http://ssrn.com/abstract=2073300>.
- Chevalier, E. (2005). Critical Price near Maturity for an American Option on a Dividend-Paying Stock in a Local Volatility Model. *Mathematical Finance*, vol. 15, no. 3, pp. 439–463.
- Chibane, M., Selvaraj, J. and Sheldon, G. (2009). Building Curves on a Good Basis. Available at: <http://ssrn.com/abstract=1394267>.
- Derman, E. and Kani, I. (1998). Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility. *International Journal of Theoretical and Applied Finance*, vol. 1, no. 01, pp. 61–110.
- Derman, E., Kani, I. and Chriss, N. (1996). Implied Trinomial Trees of the Volatility Smile. *The Journal of Derivatives*, vol. 3, no. 4, pp. 7–22.
- Doust, P. (2012). No-Arbitrage SABR. *Journal of Computational Finance*, vol. 15, no. 3, p. 3.
- Duffy, D.J. and Germani, A. (2013). *C# for Financial Markets*. John Wiley & Sons.
- Dupire, B. (1994). Pricing with a smile. *Risk Magazine*, pp. 18–20.
- Dupire, B. (1997). *Pricing and Hedging with Smiles*. Mathematics of derivative securities, edited by M.A.H. Dempster and S.R. Pliska, Cambridge University Press, Cambridge, 103–111.
- Ekstrand, C. (2011). *Financial Derivatives Modeling*. Springer.
- Errais, E., Mauri, G. and Mercurio, F. (2004). Capturing the Skew in Interest Rate Derivatives: A Shifted Lognormal LIBOR Model with Uncertain Parameters. Tech. Rep., Internal Report, Stanford University and Banca IMI.

- Fujii, M., Shimada, Y. and Takahashi, A. (2010). Collateral Posting and Choice of Collateral Currency-Implications for Derivative Pricing and Risk Management. Available at: <http://ssrn.com/abstract=1601866>.
- Fujii, M. and Takahashi, A. (2011). Choice of Collateral Currency. *Risk Magazine*, vol. 24, no. 1, pp. 120–125.
- Gatheral, J. (2006). *The Volatility Surface: A Practitioner's Guide*, vol. 357. John Wiley & Sons.
- Gatheral, J., Hsu, E.P., Laurence, P., Ouyang, C. and Wang, T.-H. (2012). Asymptotics of Implied Volatility in Local Volatility Models. *Mathematical Finance*, vol. 22, no. 4, pp. 591–620.
- Gibbs, M. and Goyder, R. (2012). The Past, Present, Future of Curves. *FINCAD*.
- Hagan, P., Lesniewski, A. and Woodward, D. (2005). Probability Distribution in the SABR Model of Stochastic Volatility. *preprint*.
- Hagan, P.S., Kumar, D., Lesniewski, A. and Woodward, D. (2014). Arbitrage-Free SABR. *Wilmott*, vol. 2014, no. 69, pp. 60–75.
- Hagan, P.S., Kumar, D., Lesniewski, A.S. and Woodward, D.E. (2002). Managing Smile Risk. *Wilmott Magazine*, July, pp. 84–108.
- Hagan, P.S. and West, G. (2006). Interpolation Methods for Curve Construction. *Applied Mathematical Finance*, vol. 13, no. 2, pp. 89–129.
- Hagan, P.S. and West, G. (2008). Methods for Constructing a Yield Curve. *Wilmott Magazine*, May, pp. 70–81.
- Hagan, P.S. and Woodward, D.E. (1999). Equivalent Black Volatilities. *Applied Mathematical Finance*, vol. 6, no. 3, pp. 147–157.
- Harrison, J.M. and Kreps, D.M. (1979). Martingales and Arbitrage in Multiperiod Securities Markets. *Journal of Economic theory*, vol. 20, no. 3, pp. 381–408.
- Harrison, J.M. and Pliska, S.R. (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic processes and their applications*, vol. 11, no. 3, pp. 215–260.
- Harrison, J.M. and Pliska, S.R. (1983). A Stochastic Calculus Model of Continuous Crading: Complete Markets. *Stochastic processes and their applications*, vol. 15, no. 3, pp. 313–316.
- Henrard, M. (2013). Multi-Curve Framework with Collateral. *OpenGamma*.
- Henry-Labordere, P. (2005). A General Asymptotic Implied Volatility for Stochastic Volatility Models. Available at: <http://ssrn.com/abstract=698601>.
- Henry-Labordere, P. (2006). Unifying the BGM and SABR Models: A short Ride in Hyperbolic Geometry. Available at: <http://ssrn.com/abstract=877762>.

- Heston, S.L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of financial studies*, vol. 6, no. 2, pp. 327–343.
- Ho, T.S. and Lee, S.B. (1986). Term Structure Movements and Pricing Interest Rate Contingent Claims. *The Journal of Finance*, vol. 41, no. 5, pp. 1011–1029.
- Hobson, D.G. and Rogers, L.C. (1998). Complete Models with Stochastic Solatility. *Mathematical Finance*, vol. 8, no. 1, pp. 27–48.
- Holmes, M.H. (2012). *Introduction to Perturbation Methods*, vol. 20. Springer.
- Hull, J. (2009). *Options, Futures and Other Derivatives*. Pearson education.
- Hull, J. and White, A. (1987). The Pricing of Options on Assets with Stochastic Volatilities. *The journal of finance*, vol. 42, no. 2, pp. 281–300.
- Hull, J. and White, A. (1993). One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities. *Journal of financial and quantitative analysis*, vol. 28, no. 02, pp. 235–254.
- Kalotay, A.J., Williams, G.O. and Fabozzi, F.J. (1993). A Model for Valuing Bonds and Embedded Options. *Financial Analysts Journal*, pp. 35–46.
- Kienitz, J. (2013). Transforming Volatility-Multi Curve Cap and Swaption Volatilities. Available at: <http://ssrn.com/abstract=2204702>.
- Lee, R.W. (2004). The Moment Formula for Implied Volatility at Extreme Strikes. *Mathematical Finance*, vol. 14, no. 3, pp. 469–480.
- Masaaki, F. and Akihiko, T. (2011). Clean Valuation Framework for the USD Silo- An Implication for the Forthcoming Standard Credit Support Annex (SCSA). Available at: <http://ssrn.com/abstract=1969686>.
- Masaaki, F., Yasufumi, S. and Akihiko, T. (2010a). Modeling of Interest Rate Term Structures Under Collateralization and its Implications. Available at: <http://ssrn.com/abstract=1681910>.
- Masaaki, F., Yasufumi, S. and Akihiko, T. (2010b). On the Term Structure of Interest Rates with Basis Spread. Available at: <http://ssrn.com/abstract=1556487>.
- Mbongo, J. (2014). Yield Curves Codes. <https://sites.google.com/a/aims.ac.za/jeffrey-mbongo/msc-code>.
- Mercurio, F. (2010). LIBOR Market Models with Stochastic Basis. *Risk Magazine*, vol. 23, no. 12, pp. 84–89.
- Mercurio, F. and Pallavicini, A. (2006). Smiling at Convexity: Bridging Swaption Skews and CMS Adjustments. Available at: <http://ssrn.com/abstract=892287>.

- Merton, R.C. (1971). Theory of rational option pricing.
- Morini, M. and Mercurio, F. (2007). No-Arbitrage Dynamics for a Tractable SABR Term Structure Libor Model. *Bloomberg Portfolio Research Paper*.
- Oblój, J. (2008). FINE-TUNE YOUR SMILE: CORRECTION to HAGAN ET AL. *Wilmott Magazine*, 35, pp. 102–104.
- Oleksandr, D., Yining, G. and Cem, Sinan, O. (2014). *Interest rates after the credit crunch crisis: SINGLE versus MULTIPLE curve approach*. Master's thesis, Barcelona GSE Master Project, <http://www.slideshare.net/barcelonagse/interest-ratescreditpaper>.
- Piterbarg, V. (2003). A Stochastic Volatility Forward Libor Model with a Term Structure of Volatility Smiles. Available at: <http://ssrn.com/abstract=472061>.
- Piterbarg, V. (2012). Cooking with collateral. *Risk Magazine*, vol. 8.
- Rebonato, R. and Rebonato, R. (1998). *Interest-Rate Option Models*, vol. 2. Wiley New York.
- Rogers, L. and Veraart, L.A. (2008). A Stochastic Volatility Alternative to SABR. *Journal of Applied Probability*, pp. 1071–1085.
- Ron, U. (2000). *A Practical Guide to Swap Curve Construction*. Bank of Canada.
- Roper, M. (2010). Arbitrage free Implied Volatility Surfaces. *preprint*.
- Shreve, S.E. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models*, vol. 11. Springer.
- Stein, E.M. and Stein, J.C. (1991). Stock Price Distributions with Stochastic Volatility: an Analytic Approach. *Review of financial Studies*, vol. 4, no. 4, pp. 727–752.
- QuantLib (Open Source). <http://quantlib.org/index.shtml>.
- West, G. (2005). Calibration of the SABR model in Illiquid Markets. *Applied Mathematical Finance*, vol. 12, no. 4, pp. 371–385.
- Wiggins, J.B. (1987). Option values under stochastic volatility: Theory and empirical estimates. *Journal of financial economics*, vol. 19, no. 2, pp. 351–372.